Scattering of massive open strings in pure spinor

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Abstract

In *Phys. Lett.* **B 660**, 583 (2008), it was proposed that the D-brane geometry could be produced by open string quantum effects. In an effort to verify the proposal, we consider scattering amplitudes involving *massive* open superstrings. The main goal of this paper is to set the ground for two-loop "renormalization" of an oriented open superstring on a D-brane and to strengthen our skill in the pure spinor formulation of a superstring, an effective tool for multi-loop string diagrams. We start by reviewing scattering amplitudes of massless states in the 2D component method of the NSR formulation. A few examples of massive string scattering are worked out. The NSR results are then reproduced in the pure spinor formulation. We compute the amplitudes using the unintegrated form of the massive vertex operator constructed by Berkovits and Chandia in *JHEP* **0208**, 040 (2002). We point out that it may be possible to discover new Riemann type identities involving Jacobi ϑ-functions by comparing a NSR computation and the corresponding pure spinor computation.

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1 Introduction

To some extent, a D-brane [1][2][3] is a peculiarity that originates from the endpoints of an open string. Before the birth of D-brane physics, an oriented open string was viewed as inconsistent because it was believed impossible to consistently couple an oriented open string to a closed string due to the different amounts of supersymmetry. With the advent of D-brane physics, open string theory has come to enjoy a more elevated status in string theory, providing a birthplace for the black hole entropy, matrix theories and AdS/CFT correspondence with its varieties. The ground breaking results just mentioned are related to the peculiarity one way or another. There may be additional new physics waiting to be discovered that is associated with the endpoints. As we describe below, the physics associated with open string divergences may be one such example.

It was long known that open string theory has divergences; they were not taken seriously before since open string theory was believed pathological anyway (for the reason stated above). Now with the elevated status of an open string, the divergence issue must be given proper consideration.[1] In a series of papers [4][5][6], the divergence issue was initiated taking the case of a D3-brane. A divergence removal procedure was proposed: the proposal may be viewed as "*renormalization*" of type IIB open superstring on a D-brane. If the picture is correct, infinitely many counter terms (or counter vertex operators to be precise) would be required, and in that sense the procedure could be taken as "renormalization" of a non-renormalizable theory. As often believed in quantum field theory, non-renormalizablity may not be an indication of a genuine pathology but rather a signal of new physics. We believe it is the case with the oriented open superstring. Although infinite in number, the counter terms may appear in a controlled manner, yielding a curved geometry when summed up. (This is in contrast to quantum field theory situation where no analogous phenomenon occurs.) In other words, the new physics may be that the D-brane curved geometry is produced by the quantum effects of open strings that are hosted by the D-brane(s). At a technical level, the generation of the curved geometry may be revealed through the renormalization.

The status of the proposal is as follows. In [6], one-loop divergence cancellation was carried out. The two-loop has been partially checked in [12]. Although the results so far are consistent with the proposal up to and including the two-loop, it is only the

[1]Our initial motivation for considering open string divergences was to ultimately derive AdS/CFT correspondence and its generalization from the first principle. For that purpose, two ingredients seem necessary. The first is the connection between geometry and open string loop effects, and the main goal of the work of [4] and the subsequent papers was to establish this connection. The other ingredient is an open string conversion into a closed string in certain circumstances. There are pieces of evidence for this [7][8][9] [10]. Once these ingredients are established, we believe that a large portion of AdS/CFT should follow from applying S-duality. Other aspects of AdS/CFT may be proven following the approach of [11]

quadratic terms[2] in the, so-called, large r_0-expansion that have played a role in the divergence cancellation mechanism. For the verification of the proposal, it is desirable to come up with an example where the higher order curvature terms do participate in the mechanism. With the hope to see the participation of the higher curvature terms, we went on to the three-loop amplitude of the massless vector states in [13]. Unlike the lower loop cases, at three-loop the amplitude itself has not been computed even in ten dimensions, i.e., the case of a spacetime filling brane. The computation was carried out using the pure spinor formulation[3] [14][15] [16][17] that was developed using ingredients of [21].

With the three-loop amplitudes within reach, one can proceed to the three-loop renormalization. However, the three-loop analysis is bound to be complex. In addition to this, it seems [13] that there is room for better understanding of the three-loop regulator of the pure spinor formulation. (See [24] for a recent related discussion.) Could there be a case that does not require a three-loop analysis and may yet unravel the role of the higher order curvature terms in the proposed renormalization? Scattering of *massive* string may be worth examining in this regard and for a few other reasons. (We will focus on three-point scattering amplitudes of massive states at tree and one-loop levels.) First of all, the higher curvature terms might begin contributing at two-loop order (and on) for the scattering of massive states.[4] (A priori, they could contribute even at one-loop. However, as we will argue later, it is unlikely to be the case.) The second reason - which is not entirely independent of the first - is that scattering of massive states might be associated with some kind of *near-extremal* geometry.[5] This is an interesting possibility to investigate. Another reason for considering the massive case is to accumulate experience in the pure spinor techniques. In the long run, it is expected that various renormalization analyses of the massless cases at two-loop and three-loop orders will be carried out in the pure spinor formulation. Therefore it will be useful to strengthen our skills with simpler exercises. Whereas massive three-point amplitudes at tree level are relatively simple (at least for bosonic states), the algebra involved in the one-loop case is comparable to that of the massless vector four-point amplitude. To assure the correctness of the results, we compute some of the amplitudes in the NSR formulation first. In general, the NSR formulation is more effective than the pure spinor formulation at tree level. At one-loop, it is so when the number of the external legs are less than or equal to four. (More remarks below on this.) As a matter of fact, the entire section 2 is devoted to the NSR formulation. Although the

[2]The quadratic terms that are referred to here are not those of the free action. They are the terms that come from expanding the curved geometry. They are different from the free action by a sign as discussed in [6].

[3]See [18][19][20] for a few related works.

[4]Scattering of some higher spin and/or massive states has been recently discussed in [22] and [23] in the NSR formulation.

[5]We thank M. M. Sheikh-Jabbari for the discussion on this point.

results are standard, we could not find any literature where, e.g., the massless vector four-point amplitude was obtained in the 2D component (as opposed to 2D superspace) NSR formulation. In the review, we collect all the necessary identities concerning Jacobi ϑ-functions and explicitly show how they can be used to simplify the intermediate expressions of amplitudes.

The rest of the paper is organized as follows. In section 2, we start by reviewing a few massless amplitudes in the NSR formulation: three-point amplitudes at tree level and the vector four-point amplitude at one-loop level. We then turn to amplitudes of massive open strings. We explicitly demonstrate use of several Riemann identities involving Jacobi ϑ-functions. Subsequently we compute the tree and one-loop amplitudes of the two massless vectors and one three-index antisymmetric tensor. As well-known, the one-loop scattering of purely massless states have the same kinematic factor as the corresponding tree amplitudes. We will see below that the same is true for amplitudes that include massive states: the one-loop amplitudes involving the first excited states have the same kinematics factor as the corresponding tree amplitudes.[6] In section 3, we reproduce the results of the three-point amplitudes in the pure spinor formulation. Both the NS and Ramond states are considered. We compute the amplitudes using the unintegrated[7] form of the massive vertex operator constructed by Berkovits and Chandia in [15]. Compared with the NSR formulation, the tree computation in the pure spinor formulation is more involved. This is also true for one-loop n-point amplitudes with $n \leq 4$. However, for a higher point amplitude, Riemann type identities involving product of five or more Jacobi ϑ-functions summed over the spin structures will be required in the NSR computation. They do not appear to be known in mathematical literature. Since the pure spinor formulation (being a variation of the Green-Schwarz formulation) does not require summing over the spin structure, it suggests a possibility that new Riemann type identities may be discovered by comparing a NSR result and the corresponding pure spinor result. Furthermore, the pure spinor formulation will be more effective in the higher loop computations, which will be needed for two-loop and three-loop open string renormalization in the near future. In section 4, we conclude with a summary and future directions. Our conventions and some useful relations are given in Appendix A and B. Part of the computation in the pure spinor formulation is presented in Appendix C.

[6]It is likely to imply that at one-loop we will not be able to see, even with the massive states, the possible role of the higher order curvature terms in the proposed mechanism of renormalization.

[7]Two-loop amplitudes require use of the integrated form of the vertex operator which is currently unavailable. We have made a substantial amout of efforts to obtain the intergrated form only to conclude that the task deserves a separate work because of the complexity.

2 Scattering of massive states in NSR formulation

In general, tree and one-loop amplitudes can be computed fairly effectively in the NSR formulation. The three- and four- point amplitudes at tree- and one- loop levels were computed long ago. However, the results are scattered in the literature. For example, the work of [25] was carried out in 2D superspace and analyzed closed string scattering. For scattering of oriented open superstrings, we could not find any NSR computation that is as explicit and direct as the computation presented below. We have decided to put together the results in the literature in a coherent manner for future purposes. In the section below, we review the tree and one-loop four-point amplitudes of the vector vertex operators. Then we compute several amplitudes that involve the three-index antisymmetric tensor of the first excited states. These amplitudes will be reproduced by using the pure spinor formulation in section 3. In the 2D component notation, a certain number of picture changing operators must be inserted. In a given amplitude, the number of picture changing operators, n_{PCO}, is given by

$$n_{PCO} = 2g - 2 + n_B + \frac{n_F}{2} \tag{1}$$

where n_B (n_F) is the number of bosons (fermions) inserted. The Mandelstam variables are defined

$$s = -(k_1 + k_2)^2 \quad t = -(k_2 + k_3)^2 \quad u = -(k_1 + k_3)^2 \tag{2}$$

The massless vector vertex operators in the (-1)- and zero- picture are given by

$$\begin{aligned} V_{A,(-1)} &= e^{-\phi}\psi^\mu e^{ik\cdot X} \\ V_{A,(0)} &= (i\partial X^\rho + 2\alpha'(k\cdot\psi)\psi^\rho)e^{ik\cdot X} \end{aligned} \tag{3}$$

and the vertex operator for the three-index anti-symmetric tensor is given by [26] [27]

$$\begin{aligned} V_{b,(-1)} &= e^{-\phi}\,(\psi^\mu\psi^\nu\psi^\kappa)e^{ik\cdot X} \\ V_{b,(0)} &= \Big[i\partial X^\mu\psi^\nu\psi^\kappa - i\partial X^\nu\psi^\mu\psi^\kappa + i\partial X^\kappa\psi^\mu\psi^\nu + (\alpha_0\cdot\psi)\psi^\mu\psi^\nu\psi^\kappa\Big]e^{ik\cdot X} \end{aligned} \tag{4}$$

2.1 review of the massless case

Below we start by reviewing the massless vector three-point amplitude. The tree-level four-point amplitude can be computed similarly as explained, e.g., in [2]. The one-loop diagrams in the NSR formulation is more complex because of the involvement of various Riemann identities. After that, we present detailed steps of the computation including all the required Riemann identities. We use the 2D component notation and employ some of the results that were obtained in [25] in the 2D superspace techniques.

massless vector three-point amplitude at tree-level

Consider the three-vector scattering at the tree level, $< V_A V_A V_A >$. We loosely denote the amplitude by $< AAA >$. Eq.(1) yields $n_{PCO} = 1$, and one possible choice of setup is to start with three picture (-1) operators and insert one PCO. Going through the procedure that is explained, e.g., in (12.5.3) and (12.5.13) of [2], one gets two (-1)-picture operators and one (0)-picture operator. The tree level correlators can be computed based on the following two-point functions,

$$\begin{aligned} < X^\mu(x) X^\nu(y) > &= -2\alpha' \eta^{\mu\nu} \ln|x-y| \\ < \psi^{\mu_1}(x) \psi^{\nu_2}(y) > &= \frac{\delta^{\mu_1\nu_2}}{x-y} \end{aligned} \tag{5}$$

One can easily show

$$\begin{aligned} & < c(x_1)c(x_2)c(x_3) >< e^{-\phi(x_1)} e^{-\phi(x_2)} > \\ & < \psi^\mu(x_1) e^{ik_1\cdot X} \psi^\nu(x_2) e^{ik_2\cdot X} (i\partial X^\rho + 2\alpha'(k_3\cdot\psi)\psi^\rho) e^{ik_3\cdot X} > \\ \to \quad & -\frac{x_{23}}{x_{12}}\,\eta^{\mu\nu} k_1^\rho - \frac{x_{13}}{x_{12}}\,\eta^{\mu\nu} k_2^\rho + \eta^{\mu\rho} k_3^\nu - \eta^{\nu\rho} k_3^\mu \end{aligned} \tag{6}$$

which, upon multiplying $\zeta_1^\mu \zeta_2^\nu \zeta_3^\rho$, yields

$$(\zeta_1\cdot\zeta_2)(k_1\cdot\zeta_3) + (\zeta_2\cdot\zeta_3)(k_2\cdot\zeta_1) + (\zeta_3\cdot\zeta_1)(k_3\cdot\zeta_2) \tag{7}$$

Permutations change only the overall numerical coefficient, and eq.(7) can be re-expressed up to an overall numerical factor as

$$\zeta_{1\mu}\zeta_{2\nu}\zeta_{3\rho} V^{\mu\nu\rho} \tag{8}$$

with

$$V^{\mu\nu\rho} \equiv \eta^{\mu\nu}(k_1^\rho - k_2^\rho) + \eta^{\nu\rho}(k_2^\mu - k_3^\mu) + \eta^{\rho\mu}(k_3^\nu - k_1^\nu)$$

The four-point tree amplitude can be similarly computed. As a matter of fact, the NSR computation is very similar to the computation in the Green-Schwarz formulation where all the vertex operators (including the ones that correspond to a bra- and a ket-states) were treated on an equal footing [6].

massless vector four-point at one-loop

The next example is massless vector four-point amplitude at one-loop.[8] The corresponding computation for closed string theory was done, e.g., in [25] using 2D superspace techniques. To make a connection with the literature such as [2], we adopt here

[8]Regarding this particular amplitude, the Green-Schwarz formulation may be the simplest. However, NSR formulation is better suited for higher n-point amplitudes or scattering of excited states as the number of inserted fields increases.

the 2D component approach. The open string four-point amplitude can be obtained by appropriately adjusting and tailoring some of the results in [25]. Most of the results can be carried over to the open string analysis with minor modifications. Imposing the boundary normal ordering as in [2], one gets for the bosonic Green's function

$$G'(x,y;\tau) = -\alpha' \ln \left| \frac{\vartheta_1(x-y,\tau)}{\vartheta'(0,\tau)} \right|^2 - \alpha' \frac{\pi}{2\tau} |x-y|^2, \tag{9}$$

where the prime on G indicates the absence of the zero modes and τ is the modulus of annulus. ϑ is a Jacobi theta function:

$$\vartheta_1(x,\tau) \equiv -\vartheta_{ab}(x,\tau) \;\text{ with }\; (a,b)=(1,1) \tag{10}$$

A few facts about Jacobi ϑ-functions are summarized in Appendix A. The fermionic two-point function is given by

$$S_\nu(x,y) \;=\; <\psi^\mu(x)\psi^\nu(y)>_\nu = \eta^{\mu\nu} \frac{\vartheta[\nu](x-y,\tau)\vartheta_1'(0,\tau)}{\vartheta[\nu](0,\tau)\vartheta_1(x-y,\tau)} \tag{11}$$

where the subscript ν represents the spin structure. For one-loop, there are four structures: "even" structures, $\nu = (0,0),(0,1),(1,0)$ and an odd structure, $\nu = (1,1)$. The four vector scattering amplitude is given by

$$\begin{aligned}
&\zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}\frac{1}{2}\int\frac{d\tau}{2\tau}\sum_\nu C_\nu <(bc)\\
&\int\prod_{i=1}^{4}dx_i(i\dot{X}^{\mu_1}+2\alpha' k_1\cdot\psi\psi^{\mu_1})e^{ik_1\cdot X(x_1)}(i\dot{X}^{\mu_2}+2\alpha' k_2\cdot\psi\psi^{\mu_2})e^{ik_2\cdot X(x_2)}\\
&(i\dot{X}^{\mu_3}+2\alpha' k_3\cdot\psi\psi^{\mu_3})e^{ik_3\cdot X(x_3)}(i\dot{X}^{\mu_4}+2\alpha' k_4\cdot\psi\psi^{\mu_4})e^{ik_4\cdot X(x_4)}>_\nu
\end{aligned} \tag{12}$$

One can take the coefficient, C_ν, as $C_{1,0} = C_{0,1} = -C_{0,0}$ [25].[9] Expansion of the matter part of the correlator in (12) yields several types of terms. The types of terms that need to be computed are[10]

$$\begin{gathered}
<XXXX>, <XX(k\psi\psi)(k\psi\psi)>\\
<X(k\psi\psi)(k\psi\psi)(k\psi\psi)>, <(k\psi\psi)(k\psi\psi)(k\psi\psi)(k\psi\psi)>
\end{gathered}$$

The first three correlators vanish for various reasons. [11] The first term trivially vanishes due to a well-known identity,

$$\sum_\nu C_\nu \vartheta_{ab}(0,\tau)^4 = 0 \tag{13}$$

[9]One need not be concerned with $C_{1,1}$: as well known, the odd structure only contributes to six- and higher- point amplitudes due to the presence of 2D fermionic zero-modes.

[10]The correlators with three X's vanishes since there is an odd number of ψ fields.

[11]The fields with the same arguments do not get contracted by way of regularization. As far as we know, this should be understood as dimensional regularization for the following reason. As two coordinates, (x,y), approach each other, the Green's functions become the same as the corresponding tree-level Green's functions. At tree-level, a Green's function with the same arguments is omitted in dimensional regularization.

One can show straightforwardly that the second and the third terms vanish as well due to the Riemann identities, (A.7). The expected one-loop result should come solely from the fourth correlator,

$$< (k\psi\psi)(k\psi\psi)(k\psi\psi)(k\psi\psi) > \tag{14}$$

Applying the standard Wick contractions produces various terms, which then get multiplied with the polarization vectors, $\zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}$, in front. Let us discuss a few examples. After some algebra, one can show that the coefficient of $(\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot \zeta_4)$ is

$$\begin{aligned}\frac{1}{4}\sum_\nu C_\nu \vartheta_{ab}(0,\tau)^4 \Big[& t^2 S_\nu(x_1-x_2)S_\nu(x_1-x_4)S_\nu(x_2-x_3)S_\nu(x_3-x_4) \\ & -u^2 S_\nu(x_1-x_2)S_\nu(x_1-x_3)S_\nu(x_2-x_4)S_\nu(x_3-x_4) \\ & +s^2 S_\nu(x_1-x_2)^2 S_\nu(x_3-x_4)^2\Big] e^{k_i\cdot k_j \ln \mathcal{F}(x_i,x_j)}\end{aligned} \tag{15}$$

where $\mathcal{F}$ is related to (9) by

$$\mathcal{F} = \ln[-G'(x,y;\tau)] \tag{16}$$

In (15), only the relevant factors among the factors present in (12) have been recorded. The factor $\vartheta_{ab}(0,\tau)^4$ arises as a result of evaluating part of the path integral as explained in Appendix A. The part in the square bracket results from the ψ-contractions. Note that the sum $\sum_\nu$ in (15) is over the even spin structures. The Riemann identity (A.9) with the fact that $\vartheta_{11}(0,\tau) = 0$ leads to

$$\begin{aligned} &\sum_{\nu=even} C_\nu \vartheta_{ab}(0,\tau)^4 S_\nu(x_1,x_2)^2 S_\nu(x_3,x_4)^2 = \vartheta_1'(0,\tau)^4 \\ &\sum_{\nu=even} C_\nu \vartheta_{ab}(0,\tau)^4 S_\nu(x_1,x_2) S_\nu(x_3,x_4) S_\nu(x_1,x_3) S_\nu(x_2,x_4) = \vartheta_1'(0,\tau)^4 \\ &\sum_{\nu=even} C_\nu \vartheta_{ab}(0,\tau)^4 S_\nu(x_1,x_2) S_\nu(x_3,x_4) S_\nu(x_1,x_4) S_\nu(x_2,x_3) = \vartheta_1'(0,\tau)^4 \end{aligned} \tag{17}$$

Using these in (15), one gets for the coefficient of $\zeta_1 \cdot \zeta_2\, \zeta_3 \cdot \zeta_4$

$$\frac{1}{4}\left[(s^2 - u^2 + t^2)\right) \tag{18}$$

The coefficients of $\zeta_1 \cdot \zeta_3\, \zeta_2 \cdot \zeta_4$ and $\zeta_1 \cdot \zeta_4\, \zeta_2 \cdot \zeta_3$ can be similarly computed: putting them together, one gets

$$\begin{aligned}\frac{1}{4}\Big[& (s^2-u^2+t^2)\zeta_1\cdot\zeta_2\,\zeta_3\cdot\zeta_4 + (u^2-s^2-t^2)\zeta_1\cdot\zeta_3\,\zeta_2\cdot\zeta_4 \\ & +(t^2-u^2+s^2)\zeta_1\cdot\zeta_4\,\zeta_2\cdot\zeta_3\Big] e^{k_i\cdot k_j \ln \mathcal{F}(x_i,x_j)}\end{aligned} \tag{19}$$

For the final forms of the coefficients, permutations of the equation above must be taken into account. Once permutations[12] are added, eq.(19) yields the expected expression,

$$\frac{1}{2}\left[tu\; \zeta_1\cdot\zeta_2\,\zeta_3\cdot\zeta_4 + st\; \zeta_1\cdot\zeta_3\,\zeta_2\cdot\zeta_4 + su\; \zeta_1\cdot\zeta_4\,\zeta_2\cdot\zeta_3\right] e^{k_i\cdot k_j \ln \mathcal{F}(x_i,x_j)} \tag{20}$$

[12] To keep the factor $e^{k_i\cdot k_j \ln F(x_i,x_j)}$ out, the dummy indices x_i's should be permuted in the same manner.

For the second example, let us consider the $(\zeta\cdot\zeta)\,(\zeta\cdot k)\,(\zeta\cdot k)$ type terms and work out $(\zeta_1\cdot\zeta_2)\,(\zeta\cdot k)\,(\zeta\cdot k)$ to be specific. Some of the terms vanish due to the Riemann identities in (A.7). The coefficient of $(\zeta_1\cdot\zeta_2)$ turns out to be

$$-\vartheta_{ab}(0,\tau)^4\Big[-\frac{s}{2}\,\Big(\zeta_3\cdot k_4\,\zeta_4\cdot k_3\Big)S_\nu(x_1,x_2)^2 S_\nu(x_3,x_4)^2 \tag{21}$$
$$+\Big(-\frac{s}{2}\,\zeta_3\!\cdot\! k_1\zeta_4\!\cdot\! k_2+\frac{u}{2}\,\zeta_3\!\cdot\! k_1\zeta_4\!\cdot\! k_3+\frac{u}{2}\,\zeta_3\!\cdot\! k_4\zeta_4\!\cdot\! k_2\Big)S_\nu(x_1,x_2)S_\nu(x_1,x_3)S_\nu(x_2,x_4)S_\nu(x_3,x_4)$$
$$+\Big(\frac{s}{2}\,\zeta_3\!\cdot\! k_2\,\zeta_4\!\cdot\! k_1-\frac{t}{2}\,\zeta_3\!\cdot\! k_2\,\zeta_4\!\cdot\! k_3-\frac{t}{2}\,\zeta_3\!\cdot\! k_4\,\zeta_4\!\cdot\! k_1\Big)S_\nu(x_1,x_2)S_\nu(x_2,x_3)S_\nu(x_1,x_4)S_\nu(x_3,x_4)\Big]$$

This result is in terms of S_ν but it can be re-expressed in terms of ϑ_ν by using the Riemann identity, (A.9)

$$-\vartheta_1'(0,\tau)^4\Big[\Big(-\frac{s}{2}\,\zeta_3\cdot k_1\zeta_4\cdot k_2+\frac{u}{2}\,\zeta_3\cdot k_1\zeta_4\cdot k_3+\frac{u}{2}\,\zeta_3\cdot k_4\zeta_4\cdot k_2\Big) \tag{22}$$
$$+\Big(\frac{s}{2}\,\zeta_3\cdot k_2\,\zeta_4\cdot k_1-\frac{t}{2}\,\zeta_3\cdot k_2\,\zeta_4\cdot k_3-\frac{t}{2}\,\zeta_3\cdot k_4\,\zeta_4\cdot k_1\Big)-\frac{s}{2}\,\Big(\zeta_3\cdot k_4\,\zeta_4\cdot k_3\Big)\Big]$$

There are five more terms of this type with different $\zeta_i\cdot\zeta_j$ in front. Taking the permutations into account and simplifying the resulting expression with the momentum conservation and transversality of the polarization vectors, one gets for the coefficient of $\zeta_1\cdot\zeta_2$

$$-\frac{1}{2}\,[\,t\,\zeta_3\cdot k_1\,\zeta_4\cdot k_2+u\,\zeta_3\cdot k_2\,\zeta_4\cdot k_1] \tag{23}$$

which is precisely the expected result as can be seen by comparing with (A.1). Collecting the results so far, one gets up to an overall numerical factor[13]

$$(2\pi)^{10}\delta(\Sigma_i k_i)K\int\frac{d\tau}{\tau^5}\int dx_1..dx_4$$
$$[\mathcal{F}(x_1,x_2)\mathcal{F}(x_3,x_4)]^{k_1\cdot k_2}\,[\mathcal{F}(x_1,x_3)\mathcal{F}(x_2,x_4)]^{k_1\cdot k_3}\,[\mathcal{F}(x_1,x_4)\mathcal{F}(x_2,x_3)]^{k_1\cdot k_4} \tag{24}$$

where the expression for kinematic factor, K, can be found in (A.1).

2.2 scattering of massive states

With the review in the previous section, the amplitudes involving massive states can be tackled.[14] In this section, we will work out a few examples. In section 3, we

[13]The additional factor of $\frac{1}{\tau^4}$ as compared with (12) appears as a result of performing integration over X-zero modes as explained in [25].

[14]In the scattering of massless states, the one-loop kinematic factors turned out to coincide with those of the corresponding tree amplitudes. (For example, in the four vector scattering both the tree amplitude and the one-loop amplitude include the factor commonly called K.) In fact, using the 2D superspace notations of [25] there is an indirect way of arguing that the bosonic coincidence (just mentioned) guarantees the coincidence in the 2D supersymmetric case, and the argument remains valid for the amplitudes involving massive states. The computations below explicitly confirm that the one-loop amplitudes have the same kinematic factors as those of the corresponding tree level amplitudes.

reproduce all the tree level amplitudes and the one-loop $< AAb >$ amplitude among the amplitudes computed here.

tree-level amplitudes of massive states

For convenience, we choose the locations of the vertex operators as

$$x_1 \to \infty, x_2 = 1, x_3 = 0 \tag{25}$$

The mass of the tensor state is given by

$$k^2 = -\frac{1}{2\alpha'} \tag{26}$$

For our first example, we consider $< V_A V_A V_b >$, which we loosely call $< AAb >$.[15] One may take the following equation as a starting point:

$$\begin{aligned} &\zeta_{\mu_1}\zeta_{\mu_2}e_{3\mu_3\nu_3\kappa_3} < c(x_1)c(x_2)c(x_3) >< e^{-\phi(x_1)}e^{-\phi(x_2)} > \\ < \Big(\psi^{\mu_1}\Big)\Big(\psi^{\mu_2}\Big)\Big[\, &i\partial X^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3} - i\partial X^{\nu_3}\psi^{\mu_3}\psi^{\kappa_3} + i\partial X^{\kappa_3}\psi^{\mu_3}\psi^{\nu_3} + (\alpha_0\cdot\psi)\psi^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3}\Big] > \\ &= -6\ \zeta_1^{\mu}\zeta_2^{\nu}k_2^{\rho}e_{3\mu\nu\rho} \end{aligned} \tag{27}$$

To this, one should add $(1 \leftrightarrow 2)$ contributions, which doubles the result. For the second example, consider $< Abb >$ amplitude. After some algebra one gets

$$\begin{aligned} &\zeta_{1\mu_1}e_{2\mu_2\nu_2\kappa_2}e_{3\mu_3\nu_3\kappa_3} < c(x_1)c(x_2)c(x_3) >< e^{-\phi(x_2)}e^{-\phi(x_3)} > \\ &< \Big[\, i\partial X^{\mu_1} + (\alpha_0\cdot\psi)\psi^{\mu_1}\Big]\left(\psi^{\mu_2}\psi^{\nu_2}\psi^{\kappa_2}\right)\left(\psi^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3}\right) > \\ = \quad &-36\ \zeta_1^{\mu}e_2^{\mu\rho_2\sigma_2}e_3^{\kappa\rho_2\sigma_2}k_1^{\kappa} \end{aligned} \tag{28}$$

As above, the contribution from permutation $(2 \leftrightarrow 3)$ should be added to this result. The final example of a three-point amplitude is $< bbb >$,

$$\begin{aligned} &e_{1\mu_1\nu_1\kappa_1}e_{2\mu_2\nu_2\kappa_2}e_{3\mu_3\nu_3\kappa_3} < c(x_1)c(x_2)c(x_3) >< e^{-\phi(x_1)}e^{-\phi(x_2)} >< \left(\psi^{\mu_1}\psi^{\nu_1}\psi^{\kappa_1}\right)\left(\psi^{\mu_2}\psi^{\nu_2}\psi^{\kappa_2}\right) \\ &\Big[\, i\partial X^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3} - i\partial X^{\nu_3}\psi^{\mu_3}\psi^{\kappa_3} + i\partial X^{\kappa_3}\psi^{\mu_3}\psi^{\nu_3} + (\alpha_0\cdot\psi)\psi^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3}\Big] > \end{aligned}$$

The result turns out to be

$$= -108\ e_1^{\rho_1\mu\nu}e_2^{\rho_2\mu\nu}e_3^{\rho_1\rho_2\kappa}k_2^{\kappa} + 96\ e_3^{\rho_1\mu\nu}e_2^{\rho_2\mu\nu}e_1^{\rho_1\rho_2\kappa}k_3^{\kappa} - 96\ e_1^{\rho_1\mu\nu}e_3^{\rho_2\mu\nu}e_2^{\rho_1\rho_2\kappa}k_3^{\kappa} \tag{29}$$

Taking permutations, $(k_2, e_2 \leftrightarrow k_3, e_3) + (k_1, e_1 \leftrightarrow k_3, e_3)$, into account, one gets

$$< bbb >= 300\,(-\ e_1^{\rho_1\mu\nu}e_2^{\rho_2\mu\nu}e_3^{\rho_1\rho_2\kappa}k_2^{\kappa} +\ e_3^{\rho_1\mu\nu}e_2^{\rho_2\mu\nu}e_1^{\rho_1\rho_2\kappa}k_3^{\kappa} -\ e_1^{\rho_1\mu\nu}e_3^{\rho_2\mu\nu}e_2^{\rho_1\rho_2\kappa}k_3^{\kappa}) \tag{30}$$

[15] As for the $< AAb >$ amplitude, the final form of the kinematic factor (i.e., $\zeta_1^{\mu}\zeta_2^{\nu}k_2^{\rho}e_{3\mu\nu\rho}$ in (27)) can be easily determined by momentum conservation and transversality of polarization tensors.

one-loop amplitudes involving massive states

As well-known, a one-loop scattering amplitude of purely massless states has the same kinematic factor as the corresponding tree amplitude. We will see below that the same is true: the one-loop amplitudes involving massive states have the same kinematics factors as those of the corresponding tree amplitudes. At one-loop, eq.(1) implies insertion of three picture changing operators: the AAb-amplitude at one-loop can be taken as

$$\zeta_1^{\mu_1}\zeta_2^{\mu_2}e_3^{\mu_3\nu_3\kappa_3}\frac{1}{2}\int\frac{dt}{2t}\sum_\nu C_\nu < (bc) \tag{31}$$
$$\int\Big(\prod_{i=1}^{3}dx_i\Big)(i\dot{X}^{\mu_1}+2\alpha' k_1\cdot\psi\psi^{\mu_1})e^{ik_1\cdot X(x_1)}(i\dot{X}^{\mu_2}+2\alpha' k_2\cdot\psi\psi^{\mu_2})e^{ik_2\cdot X(x_2)}$$
$$(\,i\partial X^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3}-i\partial X^{\nu_3}\psi^{\mu_3}\psi^{\kappa_3}+i\partial X^{\kappa_3}\psi^{\mu_3}\psi^{\nu_3}+(\alpha_0\cdot\psi)\psi^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3})e^{ik_3\cdot X(x_3)}>_\nu$$

Straightforward calculation yields

$$\begin{aligned}&\zeta_{1\mu_1}\zeta_{2\mu_2}e_{\mu_3\nu_3\rho_3}<(i\dot{X}^{\mu_1}+2\alpha' k_1\cdot\psi\psi^{\mu_1})e^{ik_1\cdot X(x_1)}(i\dot{X}^{\mu_2}+2\alpha' k_2\cdot\psi\psi^{\mu_2})e^{ik_2\cdot X(x_2)}\\&(\,i\partial X^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3}-i\partial X^{\nu_3}\psi^{\mu_3}\psi^{\kappa_3}+i\partial X^{\kappa_3}\psi^{\mu_3}\psi^{\nu_3}+(\alpha_0\cdot\psi)\psi^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3})e^{ik_3\cdot X(x_3)}>_\nu\\=\;&6k_3^2\,\zeta_1^\mu\zeta_2^\nu k_2^\rho e_{3\mu\nu\rho}S_\nu(1,3)^2S_\nu(2,3)^2\left[\mathcal{F}(z_1,z_2)\right]^{k_1\cdot k_2}\left[\mathcal{F}(z_1,z_3)\right]^{k_1\cdot k_3}\left[\mathcal{F}(z_1,z_4)\right]^{k_1\cdot k_4}\end{aligned}\tag{32}$$

Again, the function, S_ν, will not appear in the final expression. The reason is that (32), multiplies by $\vartheta_{ab}(0,\tau)^4$ and summed over the spin structures, is a special case of the first equation of the Riemann identities given in (17). Up to an overall numerical factor, one gets

$$\begin{aligned}&(2\pi)^{10}\delta(\Sigma_i k_i)(k_3^2\,\zeta_1^\mu\zeta_2^\nu k_2^\rho e_{\mu\nu\rho})\int\frac{d\tau}{\tau^5}\int dx_1dx_2dx_3\\&\left[\mathcal{F}(x_1,x_2)\right]^{k_1\cdot k_2}\left[\mathcal{F}(x_1,x_3)\right]^{k_1\cdot k_3}\left[\mathcal{F}(x_1,x_4)\right]^{k_1\cdot k_4}\end{aligned}\tag{33}$$

From now on, we will focus on the kinematics factors and not explicitly record the other factors such as $\mathcal{F}(x_i,x_j)$. Taking the spin structures into account , the $< Abb >$-amplitude is

$$\begin{aligned}&\zeta_1^{\mu_1}e_2^{\mu_2\nu_2\kappa_2}e_3^{\mu_3\nu_3\kappa_3}\frac{1}{2}\int\frac{dt}{2t}\sum_\nu C_\nu<(bc)\;\int\Big(\prod_{i=1}^{3}dx_i\Big)(i\dot{X}^{\mu_1}+2\alpha' k_1\cdot\psi\psi^{\mu_1})e^{ik_1\cdot X(x_1)}\\&(\,i\partial X^{\mu_2}\psi^{\nu_2}\psi^{\kappa_2}-i\partial X^{\nu_2}\psi^{\mu_2}\psi^{\kappa_2}+i\partial X^{\kappa_2}\psi^{\mu_2}\psi^{\nu_2}+(\alpha_0\cdot\psi)\psi^{\mu_2}\psi^{\nu_2}\psi^{\kappa_2})e^{ik_2\cdot X(x_2)}\\&(\,i\partial X^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3}-i\partial X^{\nu_3}\psi^{\mu_3}\psi^{\kappa_3}+i\partial X^{\kappa_3}\psi^{\mu_3}\psi^{\nu_3}+(\alpha_0\cdot\psi)\psi^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3})e^{ik_3\cdot X(x_3)}>_\nu\end{aligned}\tag{34}$$

The types of terms that need further consideration are $<(k_1\psi\psi)(\psi\psi)(\psi\psi\psi\psi)>$ and $<(k_1\psi\psi)(\psi\psi\psi\psi)(\psi\psi\psi\psi)>$. (The other terms will vanish due to the odd numbers of

the fermionic fields and/or dimensional regularization.) The latter turns out to vanish as we will show below but first consider the former. One can show that the correlator, $<(k_1\psi\psi)(\psi\psi)(\psi\psi\psi\psi)>$, yields

$$\begin{aligned} & \zeta_1^{\mu_1} e_2^{\mu_2\nu_2\kappa_2} e_3^{\mu_3\nu_3\kappa_3}(k_1\cdot\psi\psi^{\mu_1})(\partial X^{\mu_2}\psi^{\nu_2}\psi^{\kappa_2})(k_3\cdot\psi\ \psi^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3}) \\ \Rightarrow\quad & \partial_{z_2}G(z_2,z_1)\Big(-6\zeta_{1\mu}k_{3\mu}k_{1\nu}k_{1\beta}e_{2\nu\rho\alpha}e_{3\rho\alpha\beta}+6\zeta_{1\mu}k_{1\nu}k_{1\beta}k_{3\beta}e_{3\mu\rho\alpha}e_{2\nu\rho\alpha}\Big) \\ & +\partial_{z_2}G(z_2,z_3)\Big(-6\zeta_{1\mu}k_{3\mu}k_{3\nu}k_{1\rho}e_{2\nu\alpha\beta}e_{3\alpha\beta\rho}+6\zeta_{1\mu}k_{3\nu}k_{1\rho}k_{3\rho}e_{3\mu\alpha\beta}e_{2\nu\alpha\beta}\Big) \end{aligned} \tag{35}$$

where we have omitted the overall multiplicative factor, $S_\nu(z_1,z_3)^2S_\nu(z_2,z_3)^2$. Summing (35) and the contribution from $(e_2,k_2)\Leftrightarrow(e_3,k_3)$ given by

$$\begin{aligned} & \zeta_1^{\mu_1} e_2^{\mu_2\nu_2\kappa_2} e_3^{\mu_3\nu_3\kappa_3}(k_1\cdot\psi\psi^{\mu_1})(\partial X^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3})(k_2\cdot\psi\ \psi^{\mu_2}\psi^{\nu_2}\psi^{\kappa_2}) \\ \Rightarrow\quad & \partial_{z_2}G(z_2,z_1)\Big(-6\zeta_{1\mu}k_{2\mu}k_{1\nu}k_{1\kappa}e_{2\nu\alpha\beta}e_{3\alpha\beta\kappa}+6\zeta_{1\mu}k_{1\beta}k_{2\beta}k_{1\kappa}e_{2\mu\nu\alpha}e_{3\nu\alpha\kappa}\Big) \\ & +\partial_{z_2}G(z_2,z_3)\Big(-6\zeta_{1\mu}k_{2\mu}k_{1\nu}k_{2\kappa}e_{2\nu\rho\alpha}e_{3\rho\alpha\kappa}+6\zeta_{1\mu}k_{1\alpha}k_{2\alpha}k_{2\kappa}e_{2\mu\nu\rho}e_{3\nu\rho\kappa}\Big) \end{aligned} \tag{36}$$

one gets

$$\begin{aligned} & \partial_{z_2}G(z_2,z_1)\Big(6(k_1\cdot k_3)\zeta_{1\mu}k_{1\kappa}e_{2\kappa\alpha\beta}e_{3\mu\alpha\beta}+6(k_1\cdot k_2)\zeta_{1\mu}k_{1\kappa}e_{2\mu\alpha\beta}e_{3\kappa\alpha\beta}\Big) \\ & +\partial_{z_2}G(z_2,z_3)\Big(6(k_1\cdot k_3)\zeta_{1\mu}k_{3\kappa}e_{2\kappa\alpha\beta}e_{3\mu\alpha\beta}+6(k_1\cdot k_2)\zeta_{1\mu}k_{2\kappa}e_{2\mu\alpha\beta}e_{3\alpha\beta\kappa}\Big) \end{aligned} \tag{37}$$

Finally, Mathematica computation of $<(k_1\psi\psi)(\psi\psi\psi\psi)(\psi\psi\psi\psi)>$ yields

$$\begin{aligned} & \zeta_1^{\mu_1} e_2^{\mu_2\nu_2\kappa_2} e_3^{\mu_3\nu_3\kappa_3}(k_1\cdot\psi\psi^{\mu_1})((k_2\cdot\psi)\psi^{\mu_2}\psi^{\nu_2}\psi^{\kappa_2})((k_3\cdot\psi)\psi^{\mu_3}\psi^{\nu_3}\psi^{\kappa_3}) \\ \Rightarrow\quad & -18k_3^2\zeta_{1\mu}e_{2\mu\alpha\beta}e_{3\alpha\beta\nu}k_{1\nu}+18k_2^2\zeta_{1\mu}e_{2\nu\alpha\beta}e_{3\alpha\beta\mu}k_{1\nu}+36(\zeta_1\cdot k_2)e_{1\mu}e_{2\mu\alpha\beta}e_{3\alpha\beta\nu}k_{1\mu}k_{1\nu} \end{aligned} \tag{38}$$

The result vanishes once $(2\Leftrightarrow 3)$ contribution is added.

3 Scattering of massive states in pure spinor

In this section, we reproduce some of amplitudes computed in section 2 in the pure spinor formulation. We also compute several amplitudes that involve fermionic states. Our main goal is to set the ground for the future work where the two-loop amplitudes involving the first massive states will be computed. In the beginning we briefly review the non-minimal formulation. For the massive vertex operator of the first excited states, only the unintegrated form is known [15]. It is necessary to know the form of the integrated vertex operator as well for a general amplitude. We will take construction of the integrated vertex operator elsewhere in the near future.[16] With only the

[16]We have made preliminary attempts. Although the procedure should be straightforward in principle, the steps seem to require some intricate use of gamma matrix identities and the field equations.

unintegrated vertex operator available, there are still a few amplitudes that can be computed, and they are the focus of the current section. As we will see, the superspace description of the first excited level introduces many auxiliary fields -which is typical in a superspace formulation: gauge fixing must be proceeded before amplitude computation. We discuss below that there is a natural gauge, and the theta-expansion will be implemented in that gauge.

3.1 review of non-minimal formulation

The non-minimal version of the pure spinor formulation contains several extra fields in addition to the usual string coordinates, (X, θ): it contains the bosonic pure spinor fields, $(\lambda, \bar{\lambda})$ with their canonical conjugates, $(w, \bar{w})$ and a constrained fermionic spinor, r with its canonical conjugate, s. Each field has different number of zero modes: for the bosonic fields,

$$\begin{array}{ccccc} X^m & \lambda^\alpha & w_\alpha & \bar{\lambda}_\alpha & \bar{w}^\alpha \\ 10 & 11 & 11g & 11 & 11g \end{array} \tag{39}$$

and for the fermionic fields,

$$\begin{array}{cccc} \theta^\alpha & d_\alpha & r_\alpha & s^\alpha \\ 16 & 16g & 11 & 11g \end{array}$$

where g denotes the number of loops. Although the pure spinor formulation was formulated in the 16 component chiral notation, which is effective in most of the computations, we switch to the 32-component notation when more convenient. Manual manipulations become simpler and/or a new insight can be gained in some cases in the 32-component notation. The relation between the 16 by 16 gamma matrices, γ^m, and the 32 by 32 gamma matrices, Γ^m, is

$$\Gamma^m = \begin{pmatrix} 0 & (\gamma^m)_{\alpha\beta} \\ (\gamma^m)^{\alpha\beta} & 0 \end{pmatrix} ,$$

The pure spinor constraint, $\lambda\gamma^m\lambda = 0$, implies

$$(\lambda\gamma^m ...)(...\gamma_m\lambda) = 0 \tag{40}$$

Defining the 32-component objects

$$\lambda_u = \begin{pmatrix} \lambda^\alpha \\ 0 \end{pmatrix}, \quad \lambda_d = \begin{pmatrix} 0 \\ \lambda^\alpha \end{pmatrix} \tag{41}$$

the constraint relation above translates into

$$(\lambda_u\Gamma^m ...)(...\Gamma_m\lambda_d) = 0 \tag{42}$$

The following combination of λ-fields appears as a part of the $[ds]$-integration measure

$$(\lambda\gamma_m)_{\kappa_1}(\lambda\gamma_n)_{\kappa_2}(\lambda\gamma_p)_{\kappa_3}\gamma^{mnp}_{\kappa_4\kappa_5} \quad : \quad \text{anti-symmetric in } \kappa\text{'s} \tag{43}$$

It is is totally antisymmetric in the κ-indices. The basic OPEs [16] are

$$\begin{aligned} X^m(x)X^n(y) &= -2\,\eta^{mn}\log|x-y| \\ p_\alpha(x)\theta^\beta(y) &= \frac{\delta^\beta_\alpha}{x-y} \end{aligned} \tag{44}$$

They lead to

$$\begin{aligned} & d_\alpha\Pi_m \to \frac{\gamma^m_{\alpha\beta}}{y-z}\partial\theta^\beta,\ \Pi^m V(z) \to -\frac{2}{y-z}\,\frac{\partial}{\partial X^m}V(z) \\ & d_\alpha(y)d_\beta(z) \to -\frac{1}{y-z}\gamma^m_{\alpha\beta}\Pi_m(z),\ d_\alpha V(z) \to \frac{1}{y-z}D_\alpha V(z), \end{aligned} \tag{45}$$

where

$$\begin{aligned} d_\alpha &= p_\alpha - \frac{1}{\alpha'}\gamma^m_{\alpha\beta}\theta^\beta\partial X_m - \frac{1}{4\alpha'}\gamma^m_{\alpha\beta}\gamma_{m\rho\sigma}\theta^\beta\theta^\rho\partial\theta^\sigma \\ \Pi^m &= \partial X^m + \frac{1}{2}\theta\gamma^m\partial\theta \end{aligned} \tag{46}$$

and the covariant derivative is given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial_m \tag{47}$$

The OPEs between the currents are

$$\begin{aligned} & N_{mn}(x)\lambda^\alpha(y) \to \frac{1}{2}\frac{(\gamma_{mn}\lambda)^\alpha}{x-y}, \quad J_\lambda(x)\lambda^\alpha(y) \to \frac{\lambda^\alpha}{x-y} \\ & N^{kl}(x)N^{mn}(y) \to -3\frac{\eta^{n[k}\eta^{l]m}}{(x-y)^2} + \frac{\eta^{m[l}N^{k]n} - \eta^{n[l}N^{k]m}}{x-y} \\ & J_\lambda(x)J_\lambda(y) \to -\frac{4}{(x-y)^2}, \quad J_\lambda(x)N^{mn}(y) \to \text{regular} \end{aligned} \tag{48}$$

where

$$N_{mn} = \frac{1}{2}w\gamma_{mn}\lambda, \quad J_\lambda = w_\alpha\lambda^\alpha, \tag{49}$$

Some of the OPEs above will be used below in the amplitude computation. The unintegrated and integrated forms of the massless vertex operator are given respectively by

$$\begin{aligned} V_A &= \lambda^\alpha A_\alpha \\ U_A &= \partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + \frac{1}{2}N^{mn}\mathcal{F}_{mn} \end{aligned} \tag{50}$$

The unintegrated vertex operator for the first massive states was obtained in [15], and is given by

$$
\begin{aligned}
V_B = \quad & :\partial\theta^\beta \lambda^\alpha \gamma^{mnp}_{\alpha\beta} B_{mnp}: \\
& + \frac{1}{48} : d_\beta \lambda^\alpha (\gamma^{mnpq})^\beta{}_\alpha \partial_{[m} B_{npq]} : + \frac{3}{7} : \Pi^m \lambda^\alpha (\gamma_{st} D)_\alpha B^{st}{}_m : \\
& + \frac{1}{16} : N^{mn} \lambda^\alpha \left(3\, \partial_{[m} (\gamma_{|st|} D)_\alpha B^{st}{}_{n]} - \frac{3}{7} \partial^q (\gamma_{q[m)})^\beta{}_\alpha (\gamma_{|st|} D)_\beta B^{st}{}_{n]} \right) :
\end{aligned} \tag{51}
$$

The prescription for an arbitrary loop order has been written down. For our discussion we will need tree, one-loop and two-loop prescriptions. The prescription for computing the N-point tree amplitude is given by

$$
\mathcal{A}_{tree} =< \mathcal{N}_0(y) V_1(x_1) V_2(x_2) V_3(x_3) \int dx_4 U_4(x_4) \cdots \int dx_N U_N(x_N) > \tag{52}
$$

where

$$
\mathcal{N}_0(y) = \exp(-\lambda(y)\bar{\lambda}(y) - r(y)\theta(y)) \tag{53}
$$

is a tree-level regulator. As can be seen from above, the tree-level three-point amplitude requires only the unintegrated vertex operator. The amplitudes prescriptions for the first two loop orders are

$$
\begin{aligned}
\mathcal{A}_{1-loop} &= \int d\tau < \mathcal{N}_1(y) \int dw \mu(w) b(w) V_1(x_1) \int dx_2 U_2(x_2) \cdots \int dx_N U_N(x_N) > \\
\mathcal{A}_{2-loop} &= \int d\tau_1 d\tau_2 d\tau_3 < \mathcal{N}_2(y) \prod_{s=1}^{3} \int dw_s \mu(w_s) b(w_s) \int dx_1 U_1(x_1) \cdots \int dx_N U_N(x_N) >
\end{aligned} \tag{54}
$$

where the b ghost is a composite field given by

$$
b \;=\; s^\alpha \partial\bar{\lambda}_\alpha + \frac{\bar{\lambda}_\alpha \left[2\Pi^m (\gamma_m d)^\alpha - N_{mn} (\gamma^{mn} \partial\theta)^\alpha - J_\lambda \partial\theta^\alpha - \partial^2\theta^\alpha \right]}{4\bar{\lambda}\lambda} \tag{55}
$$

$$
+ \frac{(\bar{\lambda}\gamma^{mnp} r)(d\gamma^{mnp} d + 24 N_{mn} \Pi_p)}{192(\bar{\lambda}\lambda)^2} - \frac{(r\gamma_{mnp} r)(\bar{\lambda}\gamma^m d) N^{np}}{16(\bar{\lambda}\lambda)^3} + \frac{(r\gamma_{mnp} r)(\bar{\lambda}\gamma^{pqr} r) N^{mn} N_{qr}}{128(\bar{\lambda}\lambda)^4}
$$

For one- and two- loops, one can use the regulator given in [30]

$$
\mathcal{N}_{1,2} = e^{-\bar{\lambda}\lambda - r\theta - \bar{w}w + sd} \tag{56}
$$

The theta-expansions of the SYM fields were discussed in [31][32][33][34]. In this work, we focus on amplitudes that involve the massless vector field (and the anti-symmetric three-index tensor field for the first excited states): the ten dimensional gaugino field

is set to zero. For our computations, θ-expansion up to and including $\mathcal{O}(\theta^5)$-order is required for some fields: we have

$$
\begin{aligned}
A_\alpha &= \frac{1}{2}a_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\xi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}F_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) \\
&\quad + \frac{1}{60}(\gamma_m\theta)_\alpha(\theta\gamma^{mnp}\theta)(\partial_n\xi\gamma_p\theta) - \frac{1}{576}(\gamma^m\theta)_\alpha(\theta\gamma_m{}^{sn}\theta)(\theta\gamma_n{}^{pq}\theta)\,\partial_s\partial_q a_p + \cdots \\
A_m &= a_m - (\xi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)F_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)(\partial_n\xi\gamma_q\theta) \\
&\quad + \frac{1}{192}(\theta\gamma_m{}^{st}\theta)(\theta\gamma_t{}^{pq}\theta)\partial_s F_{pq} + \cdots \\
W^\alpha &= \xi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} + \frac{1}{48}(\gamma^{mn}\theta)^\alpha(\theta\gamma_n\gamma^{pq}\theta)\partial_m F_{pq} \\
&\quad - \frac{1}{96}(\gamma^{mn}\theta)^\alpha(\theta\gamma^{npq}\theta)\partial_m\partial_p(\xi\gamma_q\theta) - \frac{1}{1920}(\gamma^{mn}\theta)^\alpha(\theta\gamma_n{}^{st}\theta)(\theta\gamma_t{}^{pq}\theta)\partial_m\partial_s F_{pq} + \cdots \\
\mathcal{F}_{mn} &= F_{mn} - 2\partial_{[m}\xi\gamma_{n]}\theta + \frac{1}{4}(\theta\gamma_{[m}\gamma^{pq}\theta)\partial_{n]}F_{pq} - \frac{1}{6}(\theta\gamma_{[m}\gamma^{pq}\theta)\partial_{n]}\partial_p(\xi\gamma_q\theta) \\
&\quad - \frac{1}{96}(\theta\gamma_{[m}{}^{st}\theta)(\theta\gamma^{tpq}\theta)\partial_{n]}\partial_s F_{pq} + \cdots
\end{aligned}
\tag{57}
$$

The θ-expansion of the first excited states have not been written down in the literature. To implement the expansion, gauge fixing must be proceeded; we now turn to gauge fixing and the θ-expansion.

3.2 θ-expansion of the massive vertex operator

The unintegrated form of the vertex operator was obtained in [15] but without its θ-expansion. The θ-expansion of the superfield, B_{mnp}, that appears in the expression for the massive vertex operator, (51), is essential for the amplitude computation. Below we show that with a suitable gauge choice, B_{mnp} can be put into the following form:

$$
B_{mnp} = b_{mnp} - 2\,(\gamma_{[mn}\psi_{p]})_\kappa\theta^\kappa - \frac{1}{18}\,\gamma^q_{[\kappa_1\kappa_2}\left(\gamma^{[mn}\right)_{\kappa_3]}{}^\delta\partial_q\psi^{p]}_\delta\,\theta^{\kappa_1}\theta^{\kappa_2}\theta^{\kappa_3} + O(\theta^5) \tag{58}
$$

Upon substituting in (51), one gets the massive vertex operator in terms of b_{mnp} and ψ^p_α. For bosonic amplitudes, set $\psi = 0$; one gets for the vertex operator form of B_{mnp}

$$
\begin{aligned}
& B_{mnp} = b_{mnp} + O(\theta^5) \\
\Rightarrow \quad & B_{mnp} \equiv e^{mnp}\, e^{ik\cdot X} + O(\theta^5)
\end{aligned}
\tag{59}
$$

where e_{mnp} is a constant polarization tensor. All the other terms in (58) except the first term seem to contain ψ and its derivatives. Although we have checked this to the order indicated, it is likely that the full expression of V_b is

$$
B_{mnp} = e^{mnp}\, e^{ik\cdot X} \;\text{ when } \psi \text{ is set to zero} \tag{60}
$$

For amplitudes that involve both the bosons and the fermions, one should keep ψ as well in general:

$$B_{mnp} = \Big[e^{mnp} - 2\,(\gamma_{[mn}\chi_{p]})_\kappa\theta^\kappa - \frac{i}{18}\,\gamma^q_{[\kappa_1\kappa_2}\left(\gamma^{[mn}\right)_{\kappa_3]}{}^\delta k_q\chi^{p]}_\delta\,\theta^{\kappa_1}\theta^{\kappa_2}\theta^{\kappa_3}\Big]\,e^{ik\cdot X} + O(\theta^5) \tag{61}$$

where χ^p_α is a constant wave function that satisfies $k^m\chi_{m\alpha}=0$. It is also constrained constrained by $\gamma_m^{\beta\gamma}\chi^m_\gamma=0$. We consdier a few examples of those types of amplitudes toward the end of subsection 3.3.

The θ-expansion can be implemented based on the results of [15] and a gauge choice. It was stated around eq.(5.3) of [15] that various field equations can be combined to imply[17]

$$D_\alpha B^{mnp} = \gamma^{[m}_{\alpha\beta}Z^{np]\beta} - \frac{1}{48}(\gamma^{[mn})_\alpha{}^\beta H^{p]}_\beta + \gamma^{mnp}_{\alpha\gamma}Y^\gamma \tag{63}$$

H^p_β is a spin-3/2 superfield, and the precise identities of the superfields, $Z^{np\beta}$ and Y^γ do not concern us. By going to a special reference frame where the spatial momenta $k_a=0$ (the index, a, denotes the spatial directions, $a=1,...,9$), the following relations were derived,

$$\begin{aligned}&Z^{bc\gamma} = \frac{1}{4}(\gamma^{[b}\Psi^{c]})^\gamma, \quad H^b_\beta = -72\Psi^b_\beta, \quad Z^{0b\gamma} = -\frac{7}{4}\,(\gamma^0\Psi^b)^\gamma,\\ &H^0_\alpha = 0, \quad B^{0bc} = 0\end{aligned} \tag{64}$$

The spin 3/2 superfiled, Ψ^c_γ contains the physical spin 3/2 field, $\psi^c_\gamma = \Psi^c_\gamma|$, and is constrained by $\gamma_c^{\beta\gamma}\Psi^c_\gamma=0$. Upon substitution into (63), these results imply

$$D_\beta B^{abc} = 2\,(\gamma^{[ab}\Psi^{c]})_\beta \tag{65}$$

which, by applying a Lorentz transformation to a generic frame implies[18]

$$D_\beta B^{mnp} = 2\,(\gamma^{[mn}\Psi^{p]})_\beta \tag{67}$$

[17] In [15], [...] and (...) were defined without $\frac{1}{n!}$. We follow the same convention in this subsection (i.e., section 3.2). In the most part of the next subsection, however, we use a convention where

$$[...] \equiv \frac{1}{n!}(\text{anti-symmetrization}) \tag{62}$$

[18] More explicitly, consider a Lorentz transformation in the passive way. The LHS takes

$$\begin{aligned}D'_\beta B'^{mnp}(X',\theta') &= L^m_e L^n_f L^p_g\,S^\rho_\beta D_\rho B^{efg}(X,\theta)\\ &= 2\,L^m_e L^n_f L^p_g\,S^\rho_\beta(\gamma^{[ef}\Psi^{g]})_\rho(X,\theta)\\ &= 2\,(\gamma^{[mn}\Psi'^{p]})_\beta(X',\theta')\end{aligned} \tag{66}$$

where L,S denote the vector and spinor transformation matrices respectively. In the first and the third equalities, the transformation properties of fields have been used.

The θ-expansions of B_{mnp} and Ψ^c_γ can be derived from this by gauge fixing as follows. Consider the θ-expanded forms of B_{mnp} and Ψ^c_γ,

$$\begin{aligned} B^{mnp} &\equiv B^{mnp}_{(0)} + B^{mnp}_{(1)} + B^{mnp}_{(2)} + \ldots \equiv b^{mnp} + B^{mnp}_{(1)\,\kappa}\,\theta^\kappa + B^{mnp}_{(2)\,\kappa_1\kappa_2}\,\theta^{\kappa_1}\theta^{\kappa_2} + \ldots \\ \Psi^c_\gamma &\equiv \Psi^c_{(0)\gamma} + \Psi^c_{(1)\gamma} + \Psi^c_{(2)\gamma} + \ldots \equiv \psi^c_\gamma + \Psi^c_{(1)\gamma\,\kappa}\,\theta^\kappa + \Psi^c_{(2)\gamma\,\kappa_1\kappa_2}\,\theta^{\kappa_1}\theta^{\kappa_2} + \ldots \end{aligned} \tag{68}$$

Substituting these equations into (67), one gets at the first two orders of the θ-expansion[19],

$$\begin{aligned} B^{mnp}_{(1)\,\alpha} &= -2(\gamma^{[mn}\Psi^{p]}_{(0)})_\alpha \\ 2B^{mnp}_{(2)\,\alpha\beta} &= 2(\gamma^{[mn})_\alpha{}^\rho\Psi^{p]}_{(1)\,\rho\beta} - \frac{1}{2}\gamma^q_{\alpha\beta}\partial_q B^{mnp}_{(0)} \\ -3B^{mnp}_{(3)\,\alpha\kappa_1\kappa_2} &= -\frac{1}{4}\gamma^q_{\alpha[\kappa_1}\partial_q B^{mnp}_{(1)\,\kappa_2]} + 2(\gamma^{[mn})_\alpha{}^\rho\Psi^{p]}_{(1)\,\rho\kappa_1\kappa_2} \end{aligned} \tag{70}$$

Using $\Psi_{(0)} \equiv \psi$ and taking $[\alpha\beta]$ and $[\alpha\kappa_1\kappa_2]$ parts in the second and the third equations respectively, one gets

$$\begin{aligned} B^{mnp}_{(1)\,\alpha} &= -2(\gamma^{[mn}\psi^{p]})_\alpha \\ B^{mnp}_{(2)\,\alpha\beta} &= \frac{1}{2}(\gamma^{[mn})_{[\alpha}{}^\rho\Psi^{p]}_{(1)\,|\rho|\beta]} \\ -3B^{mnp}_{(3)\,\alpha\kappa_1\kappa_2} &= -\frac{1}{12}\gamma^q_{[\alpha\kappa_1}\partial_q B^{mnp}_{(1)\,\kappa_2]} + \frac{1}{3}(\gamma^{[mn})_{[\alpha}{}^\rho\Psi^{p]}_{(1)\,|\rho|\kappa_1\kappa_2]} \end{aligned} \tag{71}$$

The B-field has the usual gauge freedom, $B^{mnp} + \partial^{[m}\Lambda^{np]}$, that is associated with the field strength

$$C_{mnpq} \equiv \frac{1}{48}\partial_{[m}B_{npq]} \tag{72}$$

This freedom can be used to remove at least some of the auxiliary fields. We illustrate this with $\Psi_{(1)}$ that appears in the second and third equations in (71). As a matter of fact, $\Psi_{(1)}$ would appear in many equations that originate from comparing the higher order θ-coefficients in (67): could one set some part of $\Psi_{(1)}$ or even the whole $\Psi_{(1)}$? A desirable gauge would be the one that retains the physical spectrum, b_{mnp}, ψ^c_γ and the symmetric two-index tensor, g^{mn}, (and possibly their derivatives). The fields, b_{mnp} and ψ^c_γ, are the the zeroth order components of B_{mnp} and Ψ^c_γ respectively. As for the 44 bosonic degrees of freedom, g^{mn}, they are defined to be the zeroth component of

$$G^{mn} = D\gamma^{(m}\Psi^{n)} \tag{73}$$

[19]The convention for the covariant derivative in [15] is

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \gamma^m_{\alpha\beta}\theta^\beta\partial_m \tag{69}$$

In this paper, we use the convention of, e.g.,[16] that is quoted in (47)

From the definition of G^{mn} and the constraint (73), one gets

$$g^{mn} \equiv G^{mn}|_{\theta=0} = -\mathrm{Tr}[\gamma^{(m}\Psi^{n)}_{(1)}] \tag{74}$$

This shows that $\Psi^p_{(1)}$ contains g^{mn}: it is just the gamma-trace part of $\Psi_{(1)}$. Therefore except the gamma trace part, (74), $\Psi_{(1)}$ does not contribute to the physical content of the theory: we choose $\Lambda_{(2)}$ - which is the coefficient of the θ-quadratic term in the θ-expansion of the gauge parameter, Λ - appropriately such that

$$B^{mnp}_{(2)\,\alpha\beta} = \frac{1}{2}(\gamma^{[mn})_{[\alpha}{}^{\rho}\Psi^{p]}_{(1)\,|\rho|\beta]} = 0 \tag{75}$$

Similarly it is not difficult to see that the $\Psi_{(1)}$ part in the third equation of (71) can be removed by adjusting $\Lambda_{(3)}$ appropriately: eq.(71) is now simplified as

$$\begin{aligned} B^{mnp}_{(1)\,\alpha} &= -2(\gamma^{[mn}\psi^{p]})_{\alpha} \\ B^{mnp}_{(3)\,\alpha\beta\gamma} &= -\frac{1}{18}\gamma^{q}_{[\alpha\beta}(\gamma^{[mn}\partial_q\psi^{p]})_{\gamma]} \end{aligned} \tag{76}$$

Note that $B_{(1)}$ and $B_{(3)}$ are expressed in terms of ψ. Although we checked this for the first few orders, it seems true that one may gauge-fix Λ in such a way that $B_{(n)}$ with $n \geq 1$ would be either zero or depend on ψ. In particular, this implies

$$B_{mnp} = b_{mnp} \ \text{ when } \ \psi^c_\gamma = 0 \tag{77}$$

as indicated in (60).

3.3 amplitudes involving massive states

In this section, we compute a few examples of three-point amplitude using the result of the previous section. We compute the $< AAb >$[20] at the tree and one-loop level and confirm the results of the NSR analysis that the amplitude is proportional to the kinematic factor,[21]

$$\zeta_1^\mu \zeta_2^\nu k_2^\rho \, e_{3\mu\nu\rho}\, k_3^2 \tag{79}$$

[20] One-loop amplitudes such as $< bbb >$ or $< Abb >$ have been computed in the previous section using the NSR formulation. Reproduction of them in the pure spinor formulation must await the construction of the integrated vertex operators of the massive states.

[21] Using the momentum conservation and/or the transversality of the polarization tensors, this form can be rewritten in a few different forms. For example

$$\zeta_1^\mu \zeta_2^\nu k_2^\rho \, e_{3\mu\nu\rho}\, k_3^2 = -2\zeta_1^\mu \zeta_2^\nu k_2^\rho \, e_{3\mu\nu\rho}\, k_2 \cdot k_3 \tag{78}$$

Due to the momentum conservation and the transversality of the polarization tensors, these forms are the only forms that are allowed when there are three factors of k's. As we will see in the next section, there are five factors of k's in the case of one-loop computation. In that case as well, the kinematic factor is determined by the momentum conservation and the transversality of the polarization tensors: $(k_3^2)^2$ appears instead of k_3^2. As stated in the introduction, one of our main goals is to gain skills through simple exercises and build an "infrastructure" (such as Mathematica programming) for more complicated computations.

Towards the end, we also compute a few amplitudes that involve fermions.

3.3.1 bosonic amplitudes

$< AAb >$ tree amplitude

As in the NSR analysis, we choose the locations of the vertex operators,

$$x_1 \to \infty, x_2 = 1, x_3 = 0 \tag{80}$$

Consider $< AAb >$ amplitude,

$$< (\lambda A)(\lambda A) V_B|_{\psi=0} > \tag{81}$$

where $V_B|_{\psi=0}$ denotes the antisymmetric three-index tensor part of V_B. For convenience, we quote the vertex operator for the first excited states here again,

$$\begin{aligned} V_B|_{\psi=0} = \quad & (\lambda\gamma^{mnp}\partial\theta)\, b_{mnp} + \frac{1}{2}(d\gamma^{mnpq}\lambda)\partial_m b_{npq} + \frac{3}{7}\,\partial X^m(\lambda\gamma_{st}\gamma^l\theta)\,\partial_l\, b^{st}{}_m \\ & + \frac{1}{8}N^{mn}\left[3(\lambda\gamma_{st}\gamma^l\theta)\partial_m\partial_l\, b^{st}{}_n + \frac{3}{7}(\lambda\gamma_{qm}\gamma_{st}\gamma^l\theta)\,\partial^q\partial_l\, b^{st}{}_n\right] \end{aligned} \tag{82}$$

The first term in V_b does not contribute due to the absence of the θ-zero modes: because the zero mode function does not depend on the 2D coordinates, the term involving $\partial\theta$ would contribute only when $\partial\theta$ gets contracted with another field. As can be seen by inspection, $\partial\theta$ field does not get contracted with any other field. Let us reexpress $V_B|_{\psi=0}$ as

$$\begin{aligned} V_B|_{\psi=0} = \quad & :\partial\theta^\beta\lambda^\alpha[C(X,\theta)]_{\alpha\beta}: + :d_\beta\lambda^\alpha[E(X,\theta)]^\beta{}_\alpha: + :\partial X^m\lambda^\alpha[F(X,\theta)]_{m\alpha}: \\ & + :N^{mn}\lambda^\alpha[G(X,\theta)]_{mn\alpha}: \end{aligned} \tag{83}$$

where

$$\begin{aligned} C(X,\theta)_{\alpha\beta} &= \gamma^{mnp}_{\alpha\beta}\, b_{mnp} \\ E(X,\theta)^\beta{}_\alpha &= \frac{1}{2}(\gamma^{mnpq})^\beta{}_\alpha\partial_m b_{npq} \\ F(X,\theta)_{m\alpha} &= \frac{3}{7}\,(\gamma_{st}\gamma^l\theta)_\alpha\,\partial_l\, b^{st}{}_m \\ G(X,\theta)_{mn\alpha} &= \frac{1}{8}\left[3(\gamma_{st}\gamma^l\theta)_\alpha\partial_m\partial_l\, b^{st}{}_n + \frac{3}{7}(\gamma_{qm}\gamma_{st}\gamma^l\theta)_\alpha\,\partial^q\partial_l\, b^{st}{}_n\right] \end{aligned} \tag{84}$$

We work out $F(X,\theta)$ contribution in detail in this subsection because it is the simplest among E, F, G. The computation of $F(X,\theta)$ and $G(X,\theta)$ is placed in Appendix C.

After a few OPEs and some algebra, one can show[22]

$$
\begin{aligned}
&< \lambda A^{(1)}\, \lambda A^{(2)}\, \Pi_m \lambda^\alpha F(X,\theta)^{(3)}_{m\alpha} > && (86)\\
&= \frac{3i}{448}\Big(\zeta_1^{m_1}\zeta_2^{n_2} e_3^{stm_3} k_2^{m_2} k_2^{m_3} k_3^{q_3} < (\lambda\gamma^{m_1}\theta)(\lambda\gamma^{p_2}\theta)(\lambda\gamma_{st}\gamma^{q_3}\theta)(\theta\gamma_{m_2n_2p_2}\theta) > \\
&\qquad + \zeta_2^{m_2}\zeta_1^{n_1} e_3^{stm_3} k_1^{m_1} k_2^{m_3} k_3^{q_3} < (\lambda\gamma^{p_1}\theta)(\lambda\gamma^{m_2}\theta)(\lambda\gamma_{st}\gamma^{q_3}\theta)(\theta\gamma_{m_1n_1p_1}\theta) > \Big) \\
&\qquad\qquad + (1\leftrightarrow 2) && (87)
\end{aligned}
$$

Using the identities given in [17] (quoted in Appendix B), one gets for the first two terms in (87)

$$\zeta_1^{m_1}\zeta_2^{n_2} e_3^{stm_3} k_2^{m_2} k_2^{m_3} k_3^{q_3} < (\lambda\gamma^{m_1}\theta)(\lambda\gamma^{p_2}\theta)(\lambda\gamma_{st}\gamma^{q_3}\theta)(\theta\gamma_{m_2n_2p_2}\theta) > \; = -\frac{1}{180}\, \zeta_1^\mu \zeta_2^\nu k_2^\rho \, e_{3\mu\nu\rho}\, k_2\cdot k_3$$

$$\zeta_2^{m_2}\zeta_1^{n_1} e_3^{stm_3} k_1^{m_1} k_2^{m_3} k_3^{q_3} < (\lambda\gamma^{p_1}\theta)(\lambda\gamma^{m_2}\theta)(\lambda\gamma_{st}\gamma^{q_3}\theta)(\theta\gamma_{m_1n_1p_1}\theta) > \; = -\frac{1}{180}\zeta_1^\mu \zeta_2^\nu k_2^\rho \, e_{3\mu\nu\rho}\, k_1\cdot k_3$$

The contributions coming from $(1 \leftrightarrow 2)$ in (87) simply doubles this result.

$< AAb >$ one-loop amplitude

According to the one-loop prescription, the amplitude that we want to compute is

$$< (\lambda A)(\lambda A) V_B|_{\psi=0} >_{1-loop} \quad = \quad \int d\tau < \mathcal{N}_1(y) \int dw \mu(w) b(w) \left(\int dx_1 U_A\right)\left(\int dx_2 U_A\right) V_b > \tag{88}$$

The number of zero modes is listed in (39). At one-loop the amplitude has 16 d-zero modes. To saturate the 16 d-zero modes, the only term in (56) of the b ghost that contributes is the term that contains $(d\gamma_{mnp}d)$; the only term of the massless vector vertex operators $U(1)$ $(U(2))$ that contributes is $[d_{\alpha_1} W^{\alpha_1}_{(1)}]$ ($[d_{\alpha_2} W^{\alpha_2}_{(2)}]$); lastly the massive vertex operator contributes only through $[d_\beta \lambda^\alpha E^\beta{}_\alpha]$. Collecting these, the relevant part of the one-loop amplitude is

$$
\begin{aligned}
\mathcal{K} \;\; \equiv \;\; & \int [d\lambda][d\bar\lambda][dr][d\theta][dw][d\bar w][ds][dd] e^{-\bar\lambda\lambda - r\theta - \bar w w + sd} \\
& \frac{(\bar\lambda\gamma^{mnp}\, r)(d\gamma_{mnp}d)}{192(\bar\lambda\lambda)^2} [d_{\alpha_1} W^{\alpha_1}_{(1)}][d_{\alpha_2} W^{\alpha_2}_{(2)}][d_{\beta_3}\lambda^{\alpha_3} E^{\beta_3}_{(3)\,\alpha_3}(X,\theta)]
\end{aligned} \tag{89}
$$

[22]In the remainder of section 3.3 (and also in the appendices) we use a convention where the anti-symmetrization has unit length:

$$[...] \equiv \frac{1}{n!}(\text{permutations}) \tag{85}$$

where $\doteq$ indicates that the overall numerical coefficient is not recorded precisely. Carrying out $[ds]$ integration using the measure given in [35]

$$[ds] \;\doteq\; \frac{1}{(\lambda\bar{\lambda})^3}(\lambda\gamma^r)_{\alpha_1}(\lambda\gamma^s)_{\alpha_2}(\lambda\gamma^q)_{\alpha_3}(\gamma_{rsq})_{\alpha_4\alpha_5}\epsilon^{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}}\partial^s_{\delta_1}\cdots\partial^s_{\delta_{11}} \tag{90}$$

one gets

$$\begin{aligned}\mathcal{K} \;\doteq\; & \int[d\lambda][d\bar{\lambda}][dr][d\theta][dw][d\bar{w}][dd]e^{-\bar{\lambda}\lambda-r\theta-\bar{w}w} \\ & \frac{1}{(\lambda\bar{\lambda})^3}(\lambda\gamma^r)_{\alpha_1}(\lambda\gamma^s)_{\alpha_2}(\lambda\gamma^q)_{\alpha_3}(\gamma_{rsq})_{\alpha_4\alpha_5}\epsilon^{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}}d_{\delta_1}\cdots d_{\delta_{11}} \\ & \frac{(\bar{\lambda}\gamma^{mnp}\,r)(d\gamma_{mnp}d)}{(\bar{\lambda}\lambda)^2}[d_{\alpha_1}W^{\alpha_1}_{(1)}][d_{\alpha_2}W^{\alpha_2}_{(2)}][d_{\beta_3}\lambda^{\rho_3}E^{\beta_3}_{(3)\rho_3}(X,\theta)]\end{aligned} \tag{91}$$

Further integration over d leads to

$$\begin{aligned}\mathcal{K} \;\doteq\; & \int[d\lambda][d\bar{\lambda}][dr][d\theta][dw][d\bar{w}]e^{-\bar{\lambda}\lambda-r\theta-\bar{w}w} \\ & \frac{(\bar{\lambda}\gamma^{mnp}\,r)}{(\bar{\lambda}\lambda)^5}(\lambda\gamma^r)_{\alpha_1}(\lambda\gamma^s)_{\alpha_2}(\lambda\gamma^q)_{\alpha_3}(\gamma_{rsq})_{\alpha_4\alpha_5}\delta^{\alpha_1\ldots\alpha_5}_{\kappa_1\ldots\kappa_5} \\ & [W^{\kappa_1}_{(1)}][W^{\kappa_2}_{(2)}][\lambda^{\rho_3}E^{\kappa_3}_{(3)\rho_3}(X,\theta)](\gamma_{mnp})^{\kappa_4\kappa_5}\end{aligned} \tag{92}$$

It yields upon doing r-integration

$$\begin{aligned} & (\bar{\lambda}\gamma^{mnp}\,r)(\lambda\gamma^r)_{\alpha_1}(\lambda\gamma^s)_{\alpha_2}(\lambda\gamma^q)_{\alpha_3}(\gamma_{rsq})_{\alpha_4\alpha_5}\delta^{\alpha_1\ldots\alpha_5}_{\kappa_1\ldots\kappa_5} \\ & [W^{\kappa_1}_{(1)}][W^{\kappa_2}_{(2)}][\lambda^{\rho_3}E^{\kappa_3}_{(3)\rho_3}(X,\theta)](\gamma_{mnp})^{\kappa_4\kappa_5} \\ \doteq\; & (\bar{\lambda}\gamma_{rsq}r)(\lambda\gamma^r W_{(1)})(\lambda\gamma^s W_{(2)})(\lambda\gamma^q E_{(3)}\lambda)\end{aligned} \tag{93}$$

The freedom mentioned below (43) has been used to obtain the right-hand side. The field r can be replaced by the covariant derivative D. The covariant derivative can act either on W's or on $E_{(3)}$. When it acts on the latter, the contribution can be dropped since the result is proportional to Ψ^c_γ through the field equation and we are only considering the bosonic state setting $\Psi^c_\gamma = 0$. The covariant derivative acting on W's yields

$$\doteq (\bar{\lambda}\gamma_{rsq}\gamma^{mn}\gamma^r\lambda)\mathcal{F}_{(1)mn}(\lambda\gamma^q E_{(3)}\lambda)(\lambda\gamma^s W_{(2)}) - (\bar{\lambda}\gamma_{rsq}\gamma^{mn}\gamma^s\lambda)(\lambda\gamma^r W_{(1)})\mathcal{F}_{(2)mn}(\lambda\gamma^q E_{(3)}\lambda) \tag{94}$$

The first term of (94) yields[23]

$$
\begin{aligned}
& (\bar\lambda\gamma_{rsq}\gamma^{mn}\gamma^r\lambda)\mathcal{F}_{(1)mn}(\lambda\gamma^q E_{(3)}\lambda)(\lambda\gamma^s W_{(2)}) \\
= \quad & -\frac{i}{10}(\bar\lambda\lambda)k_1^m\zeta_1^n k_2^u k_2^s\zeta_2^q k_2^p k_3^{\mu_1}e_3^{\mu_2\mu_3\mu_4}\Big[(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma^{nuv}\theta)(\theta\gamma^{vst}\theta)(\theta\gamma^{tpq}\theta) \\
& \qquad +2\delta^n_{[u}(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma_{v]}\theta)(\theta\gamma^{vst}\theta)(\theta\gamma^{tpq}\theta)\Big] \\
& +i(\bar\lambda\lambda)k_1^s\zeta_1^t k_2^u k_2^p\zeta_2^q k_3^{\mu_1}e_3^{\mu_2\mu_3\mu_4}\Big[(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma^{nuv}\theta)k_1^{[n}(\theta\gamma^{m]st}\theta)(\theta\gamma_v{}^{pq}\theta) \\
& \qquad +2\delta^n_{[u}(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma_{v]}\theta)k_1^{[n}(\theta\gamma^{m]st}\theta)(\theta\gamma^{vpq}\theta)\Big] \\
& +\frac{i}{2}(\bar\lambda\lambda)k_1^s k_1^p\zeta_1^q k_2^u\zeta_2^v k_3^{\mu_1}e_3^{\mu_2\mu_3\mu_4}\Big[(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma^{nuv}\theta)k_1^{[n}(\theta\gamma^{m]st}\theta)(\theta\gamma_t{}^{pq}\theta) \\
& \qquad +2\delta^n_{[u}(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma_{v]}\theta)k_1^{[n}(\theta\gamma^{m]st}\theta)(\theta\gamma^{tpq}\theta)\Big]
\end{aligned} \tag{96}
$$

As with the tree-level cases, manual evaluation of these terms is tedious; we rely on the Mathematica package, Gamma.m [36]; it yields the following results:

$$
\begin{gathered}
\delta^n_{[u}(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma_{v]}\theta)(\theta\gamma^{vst}\theta)(\theta\gamma^{tpq}\theta)=0 \\
k_1^{[n}k_1^s\zeta_1^t k_2^u k_2^p\zeta_2^q k_3^{\mu_1}e_3^{\mu_2\mu_3\mu_4}\delta^n_{[u}(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma_{v]}\theta)(\theta\gamma^{m]st}\theta)(\theta\gamma^{vpq}\theta)=\frac{1}{70}(k_3^2)^2 e_{3\mu\nu\rho}k_1^\mu\zeta_1^\nu\zeta_2^\rho \\
\delta^n_{[u}(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma_{v]}\theta)(\theta\gamma^{m]st}\theta)(\theta\gamma^{tpq}\theta)=0 \\
k_1^m\zeta_1^n k_2^u k_2^s\zeta_2^p k_2^q k_3^{\mu_1}e_3^{\mu_2\mu_3\mu_4}(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma^{nuv}\theta)(\theta\gamma^{vst}\theta)(\theta\gamma^{tpq}\theta)=0 \\
k_1^{[n}k_1^s\zeta_1^t k_2^u k_2^p\zeta_2^q k_3^{\mu_1}e_3^{\mu_2\mu_3\mu_4}(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma^{nuv}\theta)(\theta\gamma^{m]st}\theta)(\theta\gamma_v{}^{pq}\theta)=\frac{1}{105}(k_3^2)^2 e_{\mu\nu\rho}k_1^\mu e_1^\nu e_2^\rho \\
k_1^{[n}k_1^s k_1^p\zeta_1^q k_2^u\zeta_2^v k_3^{\mu_1}e^{\mu_2\mu_3\mu_4}(\lambda\gamma^{m\mu_1\mu_2\mu_3\mu_4}\lambda)(\lambda\gamma^{nuv}\theta)(\theta\gamma^{m]st}\theta)(\theta\gamma_t{}^{pq}\theta)=0
\end{gathered} \tag{97}
$$

These results confirm that the kinematic factor is equivalent to (79).[24]

---

[23]Unlike the tree case, here there are five k-factors whereas there are only three factors of k's in (79). It turns out that final results contain $(k_3^2)^2$ instead of k_3^2.(It must be the case because of the momentum conservation and transversality.) It is still compatible with (79) since k_3^2 can be replaced by the on-shell value. As in the NSR computation in section 2, the three-point amplitudes contain the factor

$$[F(x_1,x_2)]^{k_1\cdot k_2}\,[F(x_1,x_3)]^{k_1\cdot k_3}\,[F(x_1,x_4)]^{k_1\cdot k_4} \tag{95}$$

which is not explicitly recorded.

[24]The $(1\leftrightarrow 2)$ contribution simply doubles this result.

3.3.2 amplitudes involving fermions

We now turn to amplitudes that involve fermionic states. In general, the NSR formulation is most effective for a tree amplitude. For a demonstration, we first compute the scattering of two gauginos and a three index tensor state. For the second example, we consider the amplitude of a vector and two spin 3/2 states.

$<\xi\xi b>$ tree and one-loop amplitudes

As in the bosonic cases, one should collect, for the tree amplitude, the terms that have five θ's among the terms that result by expanding $<(\lambda A)(\lambda A)V_B>$:

$$\begin{aligned}
&\Big[-\frac{1}{3}(\xi_1\gamma_{m_1}\theta)(\lambda\gamma^{m_1}\theta)+\frac{1}{60}(\lambda\gamma_{m_1}\theta)(\theta\gamma^{m_1n_1p_1}\theta)(\partial_{n_1}\xi_1\gamma_{p_1}\theta)\Big]\\
&\Big[-\frac{1}{3}(\xi_2\gamma_{m_2}\theta)(\lambda\gamma^{m_2}\theta)+\frac{1}{60}(\lambda\gamma_{m_2}\theta)(\theta\gamma^{m_2n_2p_2}\theta)(\partial_{n_2}\xi_2\gamma_{p_2}\theta)\Big]\\
&\Big[(\lambda\gamma^{m_3n_3p_3}\partial\theta)\,b_{m_3n_3p_3}+\frac{1}{2}(d\gamma^{m_3n_3p_3q_3}\lambda)\partial_{m_3}b_{n_3p_3q_3}+\frac{3}{7}\,\partial X^{m_3}(\lambda\gamma_{s_3t_3}\gamma^{l_3}\theta)\,\partial_{l_3}\,b^{s_3t_3}{}_{m_3}\\
&+\frac{1}{8}N^{m_3n_3}\Big(3(\lambda\gamma_{s_3t_3}\gamma^{l_3}\theta)\partial_{m_3}\partial_{l_3}\,b^{s_3t_3}{}_{n_3}+\frac{3}{7}(\lambda\gamma_{q_3m_3}\gamma_{s_3t_3}\gamma^{l_3}\theta)\,\partial^{q_3}\partial_{l_3}\,b^{s_3t_3}{}_{n_3}\Big)\Big] \qquad (98)
\end{aligned}$$

While it is possible to directly evaluate all the five-θ terms, it will be an extremely tedious task. One can save a large amount of algebra by noting that there are only a few possible kinematic factors that can be produced due to the momentum conservation, mass-shell conditions and constraints on the polarization vectors. We illustrate the idea with one of the terms that one gets by expanding (98),

$$\begin{aligned}
&(k_2)_{n_2}(k_3)_{m_3}e_{n_3p_3q_3}(\xi_1\gamma_{m_1}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma^{m_2n_2p_2}\theta)(\xi_2\gamma_{p_2}\theta)(d\gamma^{m_3n_3p_3q_3}\lambda)\\
\doteq\;&(k_2)_{n_2}(k_3)_{m_3}e_{n_3p_3q_3}(\xi_1\gamma_{m_1}\gamma^{abc}\gamma_{p_2}\xi_2)(\theta\gamma^{abc}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma^{m_2n_2p_2}\theta)(d\gamma^{m_3n_3p_3q_3}\lambda)
\end{aligned} \qquad (99)$$

where "$\doteq$" indicates that we have omitted an overall numerical factor. With this expression, one can carry out the operator product between d and θ, which produces expressions that can be further evaluated using the identities given in Appendix B. At some point of the algebra and on, it is contractions $(k_2)_{n_2}(k_3)_{m_3}e_{n_3p_3q_3}$ and one of $(\xi_1\gamma^{t_1}\xi_2),(\xi_1\gamma^{t_1t_2t_3}\xi_2)$ and $(\xi_1\gamma^{t_1\dots t_5}\xi_2)$. An inspection reveals that the only potentially non-vanishing contraction is

$$(k_2\cdot k_3)e_{t_1t_2t_3}(\xi_1\gamma^{t_1t_2t_3}\xi_2)\sim e_{t_1t_2t_3}(\xi_1\gamma^{t_1t_2t_3}\xi_2) \qquad (100)$$

where on the right hand side, momentum conservation was used and the on-shell value of k_3^2 has been omitted. The presence of this term implies an unreasonable feature that the amplitude does not depend on the orientations of the momentum vectors. We have explicitly checked for (99) (and for some of the terms in (98)) that such a term

does not survive. A similar reasoning implies that the one-loop amplitude vanishes. The fact that the one-loop vanishes is also implied by what seems to be a general feature of one-loop amplitudes: one-loop kinematic factors are the same as those of the corresponding tree amplitudes.

$< A\psi\psi >$ tree amplitude

It is possible to compute $< A\psi\psi >$ tree amplitude without knowing the integrated form of the massive vertex operator. To compute it, we consider the form of the vertex operator in (61) with $e_{mnp}=0$. (The first term in (61), e^{mnp}, will not play a role here since we consider $< A\psi\psi >$ amplitude. It would be relevant in an amplitude such as $< b\psi\psi >$.) Substituting (61) in (51), one gets

$$\begin{aligned} V_B|_{b=0} = \quad & :\partial\theta^\beta\lambda^\alpha[\tilde{C}(X,\theta)]_{\alpha\beta}: + :d_\beta\lambda^\alpha[\tilde{E}(X,\theta)]^\beta{}_\alpha: + :\Pi^m\lambda^\alpha[\tilde{F}(X,\theta)]_{m\alpha}: \\ & + :N^{mn}\lambda^\alpha[\tilde{G}(X,\theta)]_{mn\alpha}: \end{aligned} \tag{101}$$

where

$$\begin{aligned} \tilde{C}(X,\theta)_{\alpha\beta} &= \gamma^{mnp}_{\alpha\beta}\Big[-12(\gamma_{mn}\chi_p)_\kappa\theta^\kappa - 2ik_q(\gamma^q)_{\kappa_1\kappa_2}(\gamma^{mn}\chi^p)_{\kappa_3}\ \theta^{\kappa_1}\theta^{\kappa_2}\theta^{\kappa_3}\Big]e^{ik\cdot X} \\ \tilde{E}(X,\theta)^\beta{}_\alpha &= -6ik^m(\gamma^{mnpq})^\beta{}_\alpha(\gamma_{np}\chi_q)_\kappa\theta^\kappa e^{ik\cdot X} + \mathcal{O}(\theta^5) \\ \tilde{F}(X,\theta)_{m\alpha} &= \frac{18}{7}\Big[2(\gamma^{st}\gamma_{[st}\chi_{m]})_\alpha - \frac{2}{3}ik^q(\gamma_{st}\gamma^q)_{\alpha\kappa_2}(\gamma^{[st}\chi^{m]})_{\kappa_3}\ \theta^{\kappa_2}\theta^{\kappa_3} \\ &\qquad +2ik^n\,(\gamma^{st}\gamma^n)_{\alpha\beta}(\gamma_{[st}\chi_{m]})_\kappa\,\theta^\beta\theta^\kappa + \mathcal{O}(\theta^6)\Big] \\ \tilde{G}(X,\theta)_{mn\alpha} &= \frac{9}{4}\Big\{ik^m\Big[2(\gamma^{st}\gamma_{[st}\chi_{m]})_\alpha + \frac{2}{3}ik^q(\gamma_{st}\gamma^q)_{\alpha\kappa_2}(\gamma^{[st}\chi^{m]})_{\kappa_3}\theta^{\kappa_2}\theta^{\kappa_3} \\ &\qquad +2ik^n\,(\gamma^{st}\gamma^n)_{\alpha\beta}(\gamma_{[st}\chi_{m]})_\kappa\,\theta^\beta\theta^\kappa + \mathcal{O}(\theta^6)\Big] \\ &\quad +\frac{i}{7}k^p\Big[2(\gamma_{pm}\gamma^{st}\gamma_{[st}\chi_{n]})_\alpha + \frac{2}{3}ik^q(\gamma_{pm}\gamma_{st}\gamma^q)_{\alpha\kappa_2}(\gamma^{[st}\chi^{n]})_{\kappa_3}\theta^{\kappa_2}\theta^{\kappa_3} \\ &\qquad +ik^r\,(\gamma_{pm}\gamma^{st}\gamma^r)_{\alpha\beta}(\gamma_{[st}\chi_{n]})_\kappa\,\theta^\beta\theta^\kappa + \mathcal{O}(\theta^6)\Big]\Big\} \end{aligned} \tag{102}$$

The wavefunction χ satisfies

$$k^m\chi_{m\alpha} = 0, \quad \gamma^{\beta\gamma}_m\chi^m_\gamma = 0 \tag{103}$$

For the tree amplitude, one should collect the terms that contain five θ's out of the terms that result by expanding $< (\lambda A)(V_B|_{b=0})(V_B|_{b=0}) >$. Some of the resulting terms vanish for obvious reasons, and can be easily omitted. (For example, the $\tilde{C}$ term can only appear with the $\tilde{E}$ term.) The amount of the algebra involved is large even with the help of the Mathematica package Gamma.m: we will not attempt a full evaluation of $< (\lambda A)(V_B|_{b=0})(V_B|_{b=0}) >$. For an illustration we have computed the term containing $\tilde{C}\tilde{E}$ explicitly ; after lengthy and tedious manipulations, we have obtained

$$\begin{aligned} & < (\lambda A)(\partial\theta^{\beta_2}\lambda^{\alpha_2}[\tilde{C}(X,\theta)]_{\alpha_2\beta_2})(d_{\beta_3}\lambda^{\alpha_3}[\tilde{E}(X,\theta)]^{\beta_3}{}_{\alpha_3}) > \\ \doteq\ & e^m_1 k^n_1 k^p_3\Big(17\chi^{(2)}_p\gamma_n\chi^{(3)}_m - 17\chi^{(2)}_p\gamma_m\chi^{(3)}_n + 16\chi^{(2)}_n\gamma_p\chi^{(3)}_m - 16\chi^{(2)}_m\gamma_p\chi^{(3)}_n - 5\chi^{(2)}_r\gamma_m\gamma_n\gamma_p\chi^{(3)}_r\Big) \end{aligned}$$

4 Conclusion

This work is our continued effort to establish the possible connection, proposed in [4][5], between open string quantum effects and the corresponding D-brane geometry. We believe that the connection, once established, will be one of two main components that may lead to the first-principle derivation of AdS/CFT. The other component would be the conversion of an open string into a closed string discussed in [10]. The connection would not only serve in derivation of AdS/CFT but also dictate change in our notion of geometry (at least the geometry associated with a D-brane) at a fundamental level: geometry would be a secondary effect in the sense it is associated with quantum effects of gauge/open string fields.

A few checks of the proposal were carried out in [6] and [12], where divergence cancellation for scattering of massless states were analyzed at one-loop and two-loop respectively. Even though the results were consistent with the proposal, it seems necessary to go to the three-loop in order to see the involvement of the higher curvature terms. Establishing the connection, therefore, requires an effective tool for computing higher loop diagrams. We believe that the pure spinor formulation potentially provides such a tool.[25] One of the goals of this paper has been to strengthen our skills in the pure spinor formulation for its future applications in multi-loop computations. On the other hand, it is in principle possible that the role of the higher curvature terms may be revealed even at the two-loop order for scattering of *massive states*. In this paper, we have computed several scattering amplitudes that involve first massive states at tree and one-loop level, setting the grounds for two-loop computation. To assure the correctness of the results, We have carried out the analysis in the NSR formulation first, and subsequently reproduce the same results in the pure spinor formulation. The pure spinor computation requires gauge fixing, which we have discussed in section 3.2. For a general amplitude it is also necessary to construct the integrated vertex operator for the massive states, a task that deserves its own work. It would be interesting to see whether the two-loop "renormalization" (in the sense of [4][5]) of open string theory would indeed fully verify the physical picture of the open string loop-induced D-brane geometry.

There are a few other near-future directions. For the last several years, one of the active areas in AdS/CFT has been in matching the anomalous dimensions of certain SYM operators with the energies of the semi-classical configurations of a closed string [29]. With the renormalization established, it will be interesting to study how to embed the SYM analysis in a full-fledged open string analysis [28]. From our standpoint, it is rather evident that a full-fledged open string analysis should be possible. (The relevance of such an analysis is obvious but there will be more remarks on this below.) The belief is based on a few things that we discuss now. The success of comparing the planar SYM anomalous dimensions with the corresponding semi-classical closed

[25] Having an effective tool of computing superstring higher loops will be important for many applications.

string solitons seems to signify the following. In the past, there were beliefs/attempts to realize a closed string as an open string bound state. Since a closed string would not be in the Fock space, it was expected that the its realization in terms of open string fields would be complex. (The counter vertex operator proposed in [6] can be viewed as one such realization.) The complexity may be due to the fact that the close string that one attempts to realize is a *fundamental* string; if one instead considers a solitonic configuration such as the one in [29] (and many others in the related works afterwards), its construction in terms of open string fields may get vastly simplified.[26] Given that SYM is a low energy limit of an open string, the statement can be paraphrased taking the spin chain/AdS correspondence: the semi-classical closed string solitons are more complex than a fundamental closed string, and that shift of the complexity has made the corresponding gauge theory operators simpler.

Getting back to the open string realization of a closed string, it would be very surprising if the success of SYM anomalous dimensions could not be extended to the full open string description. Since the birthplace of AdS/CFT was string theory, it seems not only possible but also natural that the SYM description of a closed string soliton admits a full open string description. In addition, there is a much less understood regime of non-planar SYM. Once the large-N limit is lifted, one would have to include the entire tower of the open string massive states anyway. They would contribute to the "anomalous" dimensions of the massless modes, SYM, by circulating the loops (and the results should reduce to those of SYM by taking a low energy limit if so desired). We will report our progress on this and other related issues in the near future.

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[26] As a matter fact those kinds of semi-classical configurations may also admit a description by closed string vertex operators as discussed in [37]. The whole picture, therefore, seems to point towards the generalized open/closed string duality [10].

Appendix A: Useful identities for NSR analysis

The massless vector four-point amplitudes both at tree and one-loop levels contain the following kinematic factor,

$$
\begin{aligned}
K = \quad & -\frac{1}{4}(st\,\zeta_1\cdot\zeta_3\,\zeta_2\cdot\zeta_4 + su\,\zeta_2\cdot\zeta_3\,\zeta_1\cdot\zeta_4 + tu\,\zeta_1\cdot\zeta_2\,\zeta_3\cdot\zeta_4) \\
& +\frac{1}{2}s(\zeta_1\cdot k_4\,\zeta_3\cdot k_2\,\zeta_2\cdot\zeta_4 + \zeta_2\cdot k_3\,\zeta_4\cdot k_1\,\zeta_1\cdot\zeta_3 \\
& \qquad +\zeta_1\cdot k_3\,\zeta_4\cdot k_2\,\zeta_2\cdot\zeta_3 + \zeta_2\cdot k_4\,\zeta_3\cdot k_1\,\zeta_1\cdot\zeta_4) \\
& +\frac{1}{2}t(\zeta_2\cdot k_1\,\zeta_4\cdot k_3\,\zeta_3\cdot\zeta_1 + \zeta_3\cdot k_4\,\zeta_1\cdot k_2\,\zeta_2\cdot\zeta_4 \\
& \qquad +\zeta_2\cdot k_4\,\zeta_1\cdot k_3\,\zeta_3\cdot\zeta_4 + \zeta_3\cdot k_1\,\zeta_4\cdot k_2\,\zeta_2\cdot\zeta_1) \\
& +\frac{1}{2}u(\zeta_1\cdot k_2\,\zeta_4\cdot k_3\,\zeta_3\cdot\zeta_2 + \zeta_3\cdot k_4\,\zeta_2\cdot k_1\,\zeta_1\cdot\zeta_4 \\
& \qquad +\zeta_1\cdot k_4\,\zeta_2\cdot k_3\,\zeta_3\cdot\zeta_4 + \zeta_3\cdot k_2\,\zeta_4\cdot k_1\,\zeta_1\cdot\zeta_2)
\end{aligned} \tag{A.1}
$$

where

$$
s = -(k_1+k_2)^2 \quad t = -(k_2+k_3)^2 \quad u = -(k_1+k_3)^2 \tag{A.2}
$$

The following integral relations are useful when evaluating tree level amplitudes:

$$
\begin{aligned}
\int_0^1 dx\,\frac{1}{x}\,x^{-\alpha' s}(1-x)^{-\alpha' t} &\Rightarrow -\alpha' t \\
\int_0^1 dx\,\frac{1}{(1-x)}\,x^{-\alpha' s}(1-x)^{-\alpha' t} &\Rightarrow -\alpha' s \\
\int_0^1 dx\,\frac{1}{x(1-x)}\,x^{-\alpha' s}(1-x)^{-\alpha' t} &\Rightarrow \alpha' u \\
\int_0^1 dx\,\frac{1}{x^2}\,x^{-\alpha' s}(1-x)^{-\alpha' t} &= \frac{\alpha' t\,\alpha' u}{1+\alpha' s} \\
\int_0^1 dx\,\frac{1}{(1-x)^2}\,x^{-\alpha' s}(1-x)^{-\alpha' t} &\Rightarrow \frac{\alpha' s\,\alpha' u}{1+\alpha' t} \\
\int_0^1 dx\,x^{-\alpha' s}(1-x)^{-\alpha' t} &\Rightarrow \frac{\alpha' s\,\alpha' t}{1+\alpha' u} \\
\int_0^1 dx\,\frac{x}{(1-x)^2}\,x^{-\alpha' s}(1-x)^{-\alpha' t} &\Rightarrow \left(\alpha' s + \frac{\alpha' s\,\alpha' u}{1+\alpha' t}\right) \\
\int_0^1 dx\,\frac{1}{(1-x)^2 x}\,x^{-\alpha' s}(1-x)^{-\alpha' t} &\Rightarrow \left(\alpha' u + \frac{\alpha' s\,\alpha' u}{1+\alpha' t}\right) \\
\int_0^1 dx\,\frac{1}{(1-x)x^2}\,x^{-\alpha' s}(1-x)^{-\alpha' t} &\Rightarrow \left(\alpha' u + \frac{\alpha' t\,\alpha' u}{1+\alpha' s}\right) \\
\int_0^1 dx\,\frac{1}{x^3}\,x^{-\alpha' s}(1-x)^{-\alpha' t} &\Rightarrow \frac{(1-\alpha' u)\,\alpha' t\,\alpha' u}{(2+\alpha' s)(1+\alpha' s)}
\end{aligned} \tag{A.3}
$$

where $\Rightarrow$ indicates the fact that the following factor

$$\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} \tag{A.4}$$

has been omitted in the right hand sides. Here is a summary of our conventions for Jacobi ϑ-functions and several relations used in the NSR analysis. Our conventions for ϑ-functions are basically those of [2]. For example, the η-function is given by

$$\eta(\tau) = \left[\frac{\partial_x \vartheta_{11}(0,\tau)}{-2\pi}\right]^{1/3} \tag{A.5}$$

The prime on ϑ denotes differentiation with respect to the first argument,

$$\vartheta'(z,\tau) = \frac{\partial}{\partial x}\vartheta(x,\tau) \tag{A.6}$$

In section 2, various Riemann identities such as[27]

$$\begin{aligned}
&\sum_\nu C_\nu \vartheta_{ab}(0,\tau)^4 S_\nu(z_1-z_2)S_\nu(z_2-z_3)S_\nu(z_3-z_1) = 0\\
&\sum_\nu C_\nu \vartheta_{ab}(0,\tau)^4 S_\nu(z_1-z_2)S_\nu(z_2-z_1) = 0
\end{aligned} \tag{A.7}$$

were used. The appearance of the factor, $\vartheta_{ab}(0,\tau)^4$, can be understood as follows. It is the part of the path-integral that need to be evaluated at some point of the amplitude calculation. They basically come from the ψ- and bc- kinetic terms [25]

$$\int_{even} D(X\psi bc\beta\gamma)\left[bc\prod e^{ik_i\cdot X^i(z_i)}\right]e^{iS(X,\psi,b,c,\beta,\gamma)} \Rightarrow \frac{1}{2}\frac{\vartheta_\nu(0,\tau)^4}{\eta(\tau)^{12}} \tag{A.8}$$

where $\Rightarrow$ indicates the fact that the usual factor associated with $\mathcal{F}$ and some other irrelevant factors have been omitted. The following Riemann identity was used in several places in the NSR analysis:

$$\sum_{a,b}(-1)^{a+b}\vartheta_{ab}(x)\vartheta_{ab}(y)\vartheta_{ab}(u)\vartheta_{ab}(v) \;=\; 2\vartheta_{11}(x_1)\vartheta_{11}(y_1)\vartheta_{11}(u_1)\vartheta_{11}(v_1) \tag{A.9}$$

where

$$\begin{aligned}
x_1 &= \frac{1}{2}(x+y+u+v), \quad y_1 = \frac{1}{2}(x+y-u-v)\\
u_1 &= \frac{1}{2}(x-y+u-v), \quad v_1 = \frac{1}{2}(x-y-u+v)
\end{aligned} \tag{A.10}$$

[27]these identities can be easily derived by considering one-loop three-point amplitude and using the fact that they vanish due to the non-saturation of the fermionic zero-modes.

Appendix B: Useful identities for pure spinor computation

Our convention for the 32 by 32 gamma matrices are

$$\Gamma^m = \begin{pmatrix} 0 & (\gamma^m)_{\alpha\beta} \\ (\gamma^m)^{\alpha\beta} & 0 \end{pmatrix} ,$$

where γ^m's are the 16 by 16 gamma matrices. They satisfy[28]

$$\begin{aligned}
&\Gamma^a\Gamma_a = 10, \quad \Gamma^a\Gamma^\mu\Gamma_a = -8\Gamma^\mu, \quad \Gamma^a\Gamma^{\mu\nu}\Gamma_a = 6\Gamma^{\mu\nu} \\
&\Gamma^a\Gamma^{\mu\nu\rho}\Gamma_a = -4\Gamma^{\mu\nu\rho}, \quad \Gamma^a\Gamma^{\mu\nu\rho\sigma}\Gamma_a = 2\Gamma^{\mu\nu\rho\sigma}, \quad \Gamma^a\Gamma^{\mu\nu\rho\sigma\delta}\Gamma_a = 0 \\
&\Gamma^a\Gamma^{\mu\nu\rho\sigma\delta\kappa}\Gamma_a = -2\Gamma^{\mu\nu\rho\sigma\delta\kappa}, \quad \Gamma^a\Gamma^{\mu\nu\rho\sigma\delta\kappa\zeta}\Gamma_a = 4\Gamma^{\mu\nu\rho\sigma\delta\kappa\zeta} \\
&\Gamma^{ab}\Gamma_{ab} = -90, \quad \Gamma^{ab}\Gamma^\mu\Gamma_{ab} = -54\Gamma^\mu, \quad \Gamma^{ab}\Gamma^{\mu\nu}\Gamma_{ab} = -26\Gamma^{\mu\nu} \\
&\Gamma^{ab}\Gamma^{\mu\nu\rho}\Gamma_{ab} = -6\Gamma^{\mu\nu\rho}, \quad \Gamma^{ab}\Gamma^{\mu\nu\rho\sigma}\Gamma_{ab} = 6\Gamma^{\mu\nu\rho\sigma}, \quad \Gamma^{ab}\Gamma^{\mu\nu\rho\sigma\delta}\Gamma_{ab} = 10\Gamma^{\mu\nu\rho\sigma\delta}, \\
&\Gamma^{ab}\Gamma^{\mu\nu\rho\sigma\delta\kappa}\Gamma_{ab} = 6\Gamma^{\mu\nu\rho\sigma\delta\kappa} \\
&\Gamma^{abc}\Gamma_{abc} = -720, \quad \Gamma^{abc}\Gamma^\mu\Gamma_{abc} = 288\Gamma^\mu, \quad \Gamma^{abc}\Gamma^{\mu\nu}\Gamma_{abc} = -48\Gamma^{\mu\nu} \\
&\Gamma^{abc}\Gamma^{\mu\nu\rho}\Gamma_{abc} = -48\Gamma^{\mu\nu\rho}, \quad \Gamma^{abc}\Gamma^{\mu\nu\rho\sigma}\Gamma_{abc} = 48\Gamma^{\mu\nu\rho\sigma}, \quad \Gamma^{abc}\Gamma^{\mu\nu\rho\sigma\delta}\Gamma_{abc} = 0 \\
&\Gamma^{abc}\Gamma^{\mu\nu\rho\sigma\delta\kappa}\Gamma_{abc} = -48\Gamma^{\mu\nu\rho\sigma\delta\kappa} \\
&\Gamma^{abcd}\Gamma_{abcd} = 5040, \quad \Gamma^{abcd}\Gamma^\mu\Gamma_{abcd} = 1008\Gamma^\mu, \quad \Gamma^{abcd}\Gamma^{\mu\nu}\Gamma_{abcd} = -336\Gamma^{\mu\nu} \\
&\Gamma^{abcd}\Gamma^{\mu\nu\rho}\Gamma_{abcd} = -336\Gamma^{\mu\nu\rho}, \quad \Gamma^{abcd}\Gamma^{\mu\nu\rho\sigma}\Gamma_{abcd} = 48\Gamma^{\mu\nu\rho\sigma} \\
&\Gamma^{abcd}\Gamma^{\mu\nu\rho\sigma\delta}\Gamma_{abcd} = 240\Gamma^{\mu\nu\rho\sigma\delta}, \quad \Gamma^{abcd}\Gamma^{\mu\nu\rho\sigma\delta\kappa}\Gamma_{abcd} = 48\Gamma^{\mu\nu\rho\sigma\delta\kappa} \\
&\Gamma^{abcde}\Gamma_{abcde} = 6\cdot 5040, \quad \Gamma^{abcde}\Gamma^\mu\Gamma_{abcde} = 0, \quad \Gamma^{abcde}\Gamma^{\mu\nu}\Gamma_{abcde} = -3360\Gamma^{\mu\nu}, \\
&\Gamma^{abcde}\Gamma^{\mu\nu\rho}\Gamma_{abcde} = 0, \quad \Gamma^{abcde}\Gamma^{\mu\nu\rho\sigma}\Gamma_{abcde} = 1440\Gamma^{\mu\nu\rho\sigma} \\
&\Gamma^{abcde}\Gamma^{\mu\nu\rho\sigma\delta}\Gamma_{abcde} = 0
\end{aligned} \tag{B.1}$$

and

$$\begin{aligned}
&[\Gamma_m, \Gamma^r] = 2\Gamma_m{}^r \quad , \quad \{\Gamma_m, \Gamma^r\} = 2\delta_m{}^r \\
&\{\Gamma_{mn}, \Gamma^r\} = 2\Gamma_{mn}{}^r \quad , \quad [\Gamma_{mn}, \Gamma^r] = -4\delta^r_{[m}\Gamma_{n]} \\
&[\Gamma_{mnp}, \Gamma^r] = 2\Gamma_{mnp}{}^r \quad , \quad \{\Gamma_{mnp}, \Gamma^r\} = 6\,\delta^r_{[m}\Gamma_{np]} \\
&\{\Gamma_{mnpq}, \Gamma^r\} = 2\Gamma_{mnpq}{}^r \quad , \quad [\Gamma_{mnpq}, \Gamma^r] = -8\delta^r_{[m}\Gamma_{npq]} \\
&[\Gamma_{mnpqk}, \Gamma^r] = 2\Gamma_{mnpqk}{}^r \quad , \quad \{\Gamma_{mnpqk}, \Gamma^r\} = 10\,\delta^r_{[m}\Gamma_{npqk]} \\
&\{\Gamma_{mn}, \Gamma^{rs}\} = 2\Gamma_{mn}{}^{rs} - 4\delta^{rs}_{[mn]} \quad , \quad [\Gamma_{mn}, \Gamma^{rs}] = -8\delta^{[r}_{[m}\Gamma_{n]}{}^{s]} \\
&\{\Gamma_{mnp}\,, \Gamma^{rs}\} = 2\Gamma_{mnp}{}^{rs} - 12\delta^{rs}_{[mn}\Gamma_{p]} \quad , \quad [\Gamma_{mnp}\,, \Gamma^{rs}] = 12\delta^{[r}_{[m}\Gamma_{np]}{}^{s]} \\
&\{\Gamma_{mnpq}\,, \Gamma^{rs}\} = 2\Gamma_{mnpq}{}^{rs} - 24\delta^{rs}_{[mn}\Gamma_{pq]} \quad , \quad [\Gamma_{mnpq}\,, \Gamma^{rs}] = -16\delta^{[r}_{[m}\Gamma_{npq]}{}^{s]}
\end{aligned}$$

[28]An extensive list of identities are given here even though only a few of them were used in the actual analysis in the main body and in Appendix C. The list may serve in future computations.

$$\{\Gamma_{mnpqk}\,,\Gamma^{rs}\} = 2\Gamma_{mnpqk}{}^{rs} - 40\delta^{rs}_{[mn}\Gamma_{pqk]} \quad , \quad [\Gamma_{mnpqk}\,,\Gamma^{rs}] = 20\delta^{[r}_{[m}\Gamma_{npqk]}{}^{s]}$$
$$[\Gamma_{mnp},\Gamma^{rst}] = 2\Gamma_{mnp}{}^{rst} - 36\,\delta^{[rs}_{[mn}\Gamma_{p]}{}^{t]} \quad \{\Gamma_{mnp},\Gamma^{rst}\} = 18\delta^{[r}_{[m}\Gamma_{np]}{}^{rs]} - 12\delta^{rst}_{[mnp]}$$
$$\{\Gamma_{mnpq}\,,\Gamma^{rst}\} = 2\Gamma_{mnpq}{}^{rst} - 72\delta^{[rs}_{[mn}\Gamma_{pq]}{}^{t]}$$
$$[\Gamma_{mnpq},\Gamma^{rst}] = -24\delta^{[r}_{[m}\Gamma_{npq]}{}^{st]} + 48\,\delta^{rst}_{[mnp}\Gamma_{q]}$$
$$\{\Gamma_{mnpq}\,,\Gamma^{rstu}\} = 2\Gamma_{mnpq}{}^{rstu} - 144\delta^{rs}_{[mn}\Gamma_{pq]}{}^{tu]} + 48\delta^{rstu}_{[mnpq]} \quad ,$$
$$[\Gamma_{mnpq},\Gamma^{rstu}] = -32\delta^{[r}_{[m}\Gamma_{npq}{}^{stu]} - 64\delta^{[rst}_{[mnp}\Gamma_{q}{}^{u]}$$
$$\{\Gamma_{mnpqk},\Gamma^{rst}\} = 30\delta^{[r}_{[m}\Gamma_{npqk}{}^{st]} - 120\,\delta^{[rst}_{[mnp}\Gamma_{qk]}$$
$$[\Gamma_{mnpqk},\Gamma^{rst}] = 2\Gamma_{mnpqk}{}^{rst} - 120\,\delta^{[rs}_{[mn}\Gamma_{pqk]}{}^{t]}$$
$$\{\Gamma_{mnpqk},\Gamma^{rstu}\} = 2\Gamma_{mnpqk}{}^{rstu} - 240\,\delta^{[rs}_{[mn}\Gamma_{pqk}{}^{tu]} + 240\,\delta^{[rstu}_{[mnpq}\Gamma_{k]}$$
$$[\Gamma_{mnpqk},\Gamma^{rstu}] = 40\,\delta^{[r}_{[m}\Gamma_{npqk}{}^{stu]} - 480\,\delta^{[rst}_{[mnp}\Gamma_{qk]}{}^{u]}$$
$$\{\Gamma_{mnpqk},\Gamma^{rstuw}\} = 50\,\delta^{[r}_{[m}\Gamma_{npqk}{}^{stuw]} - 1200\,\delta^{[rst}_{[mnp}\Gamma_{qk}{}^{uw]} + 240\,\delta^{rstuw}_{mnpqk}$$
$$[\Gamma_{mnpqk},\Gamma^{rstuw}] = 2\Gamma_{mnpqk}{}^{rstuw} - 400\,\delta^{[rs}_{[mn}\Gamma_{pqk}{}^{tuw]} + 1200\,\delta^{[rstu}_{[mnpq}\Gamma_{k]}{}^{w]}$$
$$\{\Gamma_{mnpqkl},\Gamma^{r}\} = 2\Gamma_{mnpqkl}{}^{r}, \quad [\Gamma_{mnpqkl},\Gamma^{r}] = -12\,\delta^{r}_{[m}\Gamma_{npqkl]}$$
$$\{\Gamma_{mnpqkl},\Gamma^{rs}\} = 2\Gamma_{mnpqkl}{}^{rs} - 60\,\delta^{[rs}_{[mn}\Gamma_{pqkl]}, \quad [\Gamma_{mnpqkl},\Gamma^{rs}] = -24\,\delta^{[r}_{[m}\Gamma_{npqkl]}{}^{s]}$$
$$\{\Gamma_{mnpqkl},\Gamma^{rst}\} = 2\Gamma_{mnpqkl}{}^{rst} - 180\,\delta^{[rs}_{[mn}\Gamma_{pqkl]}{}^{t]}$$
$$[\Gamma_{mnpqkl},\Gamma^{rst}] = -36\delta^{[r}_{[m}\Gamma_{npqkl}{}^{st]} + 240\,\delta^{rst}_{[mnp}\Gamma_{qkl]}$$
$$\{\Gamma_{mnpqkl},\Gamma^{rstu}\} = 2\Gamma_{mnpqkl}{}^{rstu} - 360\,\delta^{[rs}_{[mn}\Gamma_{pqkl}{}^{tu]} + 720\,\delta^{rstu}_{[mnpq}\Gamma_{kl]}$$
$$[\Gamma_{mnpqkl},\Gamma^{rstu}] = -48\,\delta^{[r}_{[m}\Gamma_{npqkl}{}^{stu]} + 960\,\delta^{[rst}_{[mnp}\Gamma^{u]}_{qkl]}$$
$$\{\Gamma_{mnpqkl},\Gamma^{rstuw}\} = -600\,\delta^{[rs}_{[mn}\Gamma_{pqkl}{}^{tuw]} + 3600\,\delta^{[rstu}_{[mnpq}\Gamma_{kl]}{}^{w]}$$
$$[\Gamma_{mnpqkl},\Gamma^{rstuw}] = -60\,\delta^{[r}_{[m}\Gamma_{npqkl}{}^{stuw]} + 2400\,\delta^{[rst}_{[mnp}\Gamma_{qkl}{}^{uw]} - 1440\,\delta^{rstuw}_{[mnpqk}\Gamma_{l]}$$

The following relations were used when a transpose of the 32 by 32 gamma matrix was taken,

$$\begin{aligned}
(\Gamma^{\mu})^T &= \Gamma^0\Gamma^{\mu}\Gamma^0 \\
(\Gamma^{\mu\nu})^T &= \Gamma^0\Gamma^{\mu\nu}\Gamma^0 \\
(\Gamma^{\mu\nu\rho})^T &= -\Gamma^0\Gamma^{\mu\nu\rho}\Gamma^0 \\
(\Gamma^{\mu\nu\rho\sigma})^T &= -\Gamma^0\Gamma^{\mu\nu\rho\sigma}\Gamma^0 \\
(\Gamma^{\mu_1\cdots\mu_5})^T &= \Gamma^0\Gamma^{\mu_1\cdots\mu_5}\Gamma^0
\end{aligned} \tag{B.2}$$

Some of the identities given in [17] were used in the computations in section 3.3. To make this paper self-contained we present them below:

$$< (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{ijk}\theta) > \;=\; \frac{1}{120}\delta^{mnp}_{ijk}$$

$$< (\lambda\gamma^{mnp}\theta)(\lambda\gamma_q\theta)(\lambda\gamma_t\theta)(\theta\gamma_{ijk}\theta) > \;=\; \frac{1}{70}\delta_{[q}^{[m}\eta_{t][i}\delta_j^n\delta_{k]}^{p]} \qquad \text{(B.3)}$$

$$\begin{aligned} & < (\lambda\gamma_t\theta)(\lambda\gamma^{mnp}\theta)(\lambda\gamma^{qrs}\theta)(\theta\gamma_{ijk}\theta) > \\ = \quad & \frac{1}{8400}\epsilon^{ijkmnpqrst} + \frac{1}{140}\Big(\delta_t^{[m}\delta_{[i}^n\eta^{p][q}\delta_j^r\delta_{k]}^{s]} - \delta_t^{[q}\delta_{[i}^r\eta^{s][m}\delta_j^n\delta_{k]}^{p]}\Big) \\ & -\frac{1}{280}\Big(\eta_{t[i}\eta^{v[q}\delta_j^r\eta^{s][m}\delta_{k]}^n\delta_v^{p]} - \eta_{t[i}\eta^{v[m}\delta_j^n\eta^{p][q}\delta_{k]}^r\delta_v^{s]}\Big) \end{aligned} \qquad \text{(B.4)}$$

$$\begin{aligned} & < (\lambda\gamma_{mnpqr}\lambda)(\lambda\gamma^u\theta)(\theta\gamma^{fgh}\theta)(\theta\gamma_{jkl}\theta) > \\ = \quad & -\frac{4}{35}\Big(\delta_{[j}^{[m}\delta_k^n\delta_{l]}^p\delta_{[f}^q\delta_g^{r]}\delta_{h]}^u + \delta_{[f}^{[m}\delta_g^n\delta_{h]}^p\delta_{[j}^q\delta_k^{r]}\delta_{l]}^u - \frac{1}{2}\delta_{[j}^{[m}\delta_k^n\eta_{l][f}\delta_g^p\delta_{h]}^q\eta^{r]u} - \frac{1}{2}\delta_{[f}^{[m}\delta_g^n\eta_{h][j}\delta_k^p\delta_{l]}^q\eta^{r]u}\Big) \\ & -\frac{1}{1050}\epsilon^{mnpqr}{}_{abcde}\Big(\delta_{[j}^{[a}\delta_k^b\delta_{l]}^c\delta_{[f}^d\delta_g^{e]}\delta_{h]}^u + \delta_{[f}^{[a}\delta_g^b\delta_{h]}^c\delta_{[j}^d\delta_k^{e]}\delta_{l]}^u \\ & \qquad -\frac{1}{2}\delta_{[j}^{[a}\delta_k^b\eta_{l][f}\delta_g^c\delta_{h]}^d\eta^{e]u} - \frac{1}{2}\delta_{[f}^{[a}\delta_g^b\eta_{h][j}\delta_k^c\delta_{l]}^d\eta^{e]u}\Big) \end{aligned} \qquad \text{(B.5)}$$

$$\begin{aligned} & < (\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_{stu}\theta)(\lambda\gamma_v\theta)(\theta\gamma_{fgh}\theta) > \\ = \quad & \frac{1}{35}\eta^{v[m}\delta_{[s}^n\delta_t^p\eta_{u][f}\delta_g^q\delta_{h]}^{r]} - \frac{2}{35}\delta_{[s}^{[m}\delta_t^n\delta_{u]}^p\delta_{[f}^q\delta_g^{r]}\delta_{h]}^v \\ & +\frac{1}{120}\epsilon^{mnpqr}{}_{abcde}\Big(\frac{1}{35}\eta^{v[a}\delta_{[s}^b\delta_t^c\eta_{u][f}\delta_g^d\delta_{h]}^{e]} - \frac{2}{35}\delta_{[s}^{[a}\delta_t^b\delta_{u]}^c\delta_{[f}^d\delta_g^{e]}\delta_{h]}^v\Big) \end{aligned} \qquad \text{(B.6)}$$

$$< (\lambda\gamma^{mnpqr}\theta)(\lambda\gamma_d\theta)(\lambda\gamma_e\theta)(\theta\gamma_{fgh}\theta) >= -\frac{1}{42}\delta_{abcde}^{mnpqr} - \frac{1}{5040}\epsilon^{mnpqr}{}_{defgh} \qquad \text{(B.7)}$$

Appendix C: Contribution of $N^{mn}\lambda^\alpha G(X,\theta)_{mn\alpha}$ to AAb

In section 3.3, we computed the contribution of F in (83) to AAB-amplitude at tree level. Here we present the contributions of the terms that contain,

$$\begin{aligned} E(X,\theta)^\beta{}_\alpha &= \frac{1}{2}(\gamma^{mnpq})^\beta{}_\alpha\partial_m b_{npq} \\ G(X,\theta)_{mn\alpha} &= \frac{1}{8}\Big[3(\gamma_{st}\gamma^l\theta)_\alpha\partial_m\partial_l\, b^{st}{}_n + \frac{3}{7}(\gamma_{qm}\gamma_{st}\gamma^l\theta)_\alpha\,\partial^q\partial_l\, b^{st}{}_n\Big] \end{aligned} \qquad \text{(C.1)}$$

Following the steps similar to those of section 3.3, one can show

$$\begin{aligned} & < \lambda A^{(1)}\,\lambda A^{(2)}\, d_\beta\lambda^\alpha[E(X,\theta)^{(3)}]^\beta{}_\alpha > \\ = \quad & \frac{i}{96\cdot 576}\zeta_1^m\zeta_2^{p'}e_{3npq}k_2^s k_2^{q'}k_3^m \\ & \Big[< (\lambda\gamma^{m_2}\gamma^{mnpq}\lambda)(\lambda\gamma^{m_1}\theta)(\theta\gamma_{m_2}{}^{sn_2}\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\ & -2 < (\lambda\gamma^{mnpq}\gamma^{m_2sn_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\ & -2 < (\lambda\gamma^{mnpq}\gamma^{n_2p'q'}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{m_2}{}^{sn_2}\theta) > \Big] + (1\leftrightarrow 2) \end{aligned} \qquad \text{(C.2)}$$

The term of $(1 \leftrightarrow 2)$ entirely vanishes according to Mathematica computation; so does the first term in (C.2). Using the gamma matrix identities given in Appendix B, the second term in (C.2) can be rewritten as

$$\begin{aligned}
& \zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m < (\lambda\gamma^{mnpq}\gamma^{m_2 s n_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
= \quad & \zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m \\
& \Big(< (\lambda\gamma_{mnpq}\gamma^{m_2}\gamma^{sn_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
& - < (\lambda\gamma_{mnpqn_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_s\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
& +4 < (\lambda\delta^{n_2}_{[m}\gamma_{npq]}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_s\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
& + < (\lambda\gamma_{mnpqs}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_s\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
& -4 < (\lambda\delta^{s}_{[m}\gamma_{npq]}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{n_2}\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \Big)
\end{aligned} \tag{C.3}$$

It turns out that only the first term yields a non-vanishing result. The first term can be rewritten as

$$\begin{aligned}
& \zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m < (\lambda\gamma_{mnpq}\gamma^{m_2}\gamma^{sn_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
= \quad & 6\,\zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m \Big(< (\lambda\gamma_{mpqsn_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_n\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
& \qquad -6 < (\lambda\delta^{sn_2}_{[mp}\gamma_{q]}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_n\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
& \qquad +6 < (\lambda\delta^{s}_{[m}\gamma_{pq]}{}^{n_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_n\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \Big) \\
& -2\,\zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m \Big(< (\lambda\gamma_{npqsn_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_m\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
& \qquad -6 < (\lambda\delta^{sn_2}_{[np}\gamma_{q]}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_m\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
& \qquad +6 < (\lambda\delta^{s}_{[n}\gamma_{pq]}{}^{n_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_m\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \Big)
\end{aligned} \tag{C.4}$$

Each of the term in (C.4) is analyzed with the following results:

$$\zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m < (\lambda\gamma_{mpqsn_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_n\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) >= -\frac{1}{420}\zeta_1^\mu \zeta_2^\nu k_2^\rho e_{3\mu\nu\rho}\, k_2 \cdot k_3$$

$$\zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m < (\lambda\delta^{sn_2}_{[mp}\gamma_{q]}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_n\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) >= -\frac{1}{2160}\,\zeta_1^\mu \zeta_2^\nu k_2^\rho e_{3\mu\nu\rho}\, k_2 \cdot k_3$$

$$\zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m < (\lambda\delta^{s}_{[m}\gamma_{pq]}{}^{n_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_n\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) >= -\frac{1}{1512}\,\zeta_1^\mu \zeta_2^\nu k_2^\rho e_{3\mu\nu\rho}\, k_2 \cdot k_3$$

$$\zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m < (\lambda\gamma_{npqsn_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_m\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) >= -\frac{1}{140}\zeta_1^\mu \zeta_2^\nu k_2^\rho e_{3\mu\nu\rho}\, k_2 \cdot k_3$$

$$\zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m < (\lambda\delta^{sn_2}_{[np}\gamma_{q]}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_m\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) >= -\frac{1}{720}\zeta_1^\mu \zeta_2^\nu k_2^\rho e_{3\mu\nu\rho}\, k_2 \cdot k_3$$

$$\zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m < (\lambda\delta^{s}_{[n}\gamma_{pq]}{}^{n_2}\theta)(\lambda\gamma^{m_1}\theta(\lambda\gamma_m\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) = -\frac{1}{180}\zeta_1^\mu \zeta_2^\nu k_2^\rho e_{3\mu\nu\rho}\, k_2 \cdot k_3$$

Combining these, the first term of (C.3) becomes

$$\zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m < (\lambda\gamma_{mnpq}\gamma^{m_2}\gamma^{sn_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) >= \frac{3}{70}\zeta_1^\mu \zeta_2^\nu k_2^\rho e_{3\mu\nu\rho}\, k_2{\cdot}k_3$$

The third term in (C.2)

$$\begin{aligned}
& \zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m < (\lambda\gamma^{mnpq}\gamma^{m_2sn_2}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{n_2}{}^{p'q'}\theta) > \\
= \quad & \zeta_1^m \zeta_2^{p'} e_{3npq} k_2^s k_2^{q'} k_3^m \\
& \Big(-\frac{1}{3} < (\lambda\gamma_{mnpqn_2p'q'}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{m_2}{}^{sm_2}\theta) > \\
& -36 < (\lambda\delta^{n_2p'}_{[mn}\gamma_{pq]}{}^{q'}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{m_2}{}^{sn_2}\theta) > \\
& -12 < (\lambda\delta^{n_2}_{[m}\gamma_{npq]}{}^{p'q'}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{m_2}{}^{sn_2}\theta) > \\
& +24 < (\lambda\delta^{n_2p'q'}_{[mnp}\gamma_{q]}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma_{m_2}\theta)(\theta\gamma_{m_2}{}^{sn_2}\theta) > \Big) \qquad \text{(C.5)}
\end{aligned}$$

After some algebra, it was found that all of the four terms in (C.5) vanish. As announced above (79), the overall result of the tree amplitude shows that the kinematic factor is the same as (79) up to a numerical constant. The amplitude from the fourth term of (83) is

$$\begin{aligned}
& < (\lambda A^{(1)})\,(\lambda A^{(2)})\,N^{mn}\lambda^\alpha G(X,\theta)^{(3)}_{mn\alpha} > \qquad \text{(C.6)} \\
= \quad & < \frac{1}{2}\frac{(\gamma_{mn}\lambda)^{\alpha_1}}{z_3 - z_1} A^{(1)}_{\alpha_1}\,(\lambda A^{(2)})\lambda^\alpha G(X,\theta)^{(3)}_{mn\alpha} > + < (\lambda A^{(1)})\frac{1}{2}\frac{(\gamma_{mn}\lambda)^{\alpha_2}}{z_3 - z_2} A^{(2)}_{\alpha_2}\,\lambda^\alpha G(X,\theta)^{(3)}_{mn\alpha} >
\end{aligned}$$

where the equality is obtained by applying the operator product expansion of λ and N^{mn}. As in section 3, we choose $x_1 = \infty, x_2 = 1, x_3 = 0$; the first term vanishes. Explicitly substituting the expression for $G(X,\theta)_{mn\alpha}$, one gets

$$= \quad \frac{1}{8} < (\lambda A^{(1)})(\lambda\gamma_{mn}A^{(2)})\Big[3(\lambda\gamma_{st}\gamma^l\theta)k_3^l k_3^m e_3^{stn} + \frac{3}{7}(\lambda\gamma_{qm}\gamma_{st}\gamma^l\theta)k_3^q k_3^l e_3^{stn}\Big] > \qquad \text{(C.7)}$$

Upon substituting (57) in the equation above, the first term of (C.7) yields

$$\begin{aligned}
& \frac{1}{8} < (\lambda A^{(1)})(\lambda\gamma_{mn}A^{(2)})(\lambda\gamma_{st}\gamma^l\theta)k_3^l k_3^m e_3^{stn} > \\
= \quad & -\frac{3i}{16^2}\,\zeta_1^{m_1}\zeta_2^{n_2} e_3^{stn} k_2^{m_2} k_3^l k_3^m < (\lambda\gamma^{st}\gamma^l\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma^{mn}\gamma^{p_2}\theta)(\theta\gamma^{m_2n_2p_2}\theta) > \\
& -\frac{3i}{16^2}\,\zeta_1^{n_1}\zeta_2^{m_2} e_3^{stn} k_1^{m_1} k_3^l k_3^m < (\lambda\gamma^{st}\gamma^l\theta)(\lambda\gamma^{p_1}\theta)(\lambda\gamma^{mn}\gamma^{m_2}\theta)(\theta\gamma^{m_1n_1p_1}\theta) > \qquad \text{(C.8)}
\end{aligned}$$

Using one of the gamma matrix identities, the first term of (C.8) (omitting the overall numerical coefficient) can be re-expressed as

$$\zeta_1^{m_1}\zeta_2^{n_2} e_3^{stn} k_2^{m_2} k_3^l k_3^m < (\lambda\gamma^{st}\gamma^l\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma^{mn}\gamma^{p_2}\theta)(\theta\gamma^{m_2n_2p_2}\theta) >$$

$$
\begin{aligned}
= \quad & \zeta_1^{m_1}\zeta_2^{n_2}e_3^{stn}k_2^{m_2}k_3^l k_3^m \Big(< (\lambda\gamma^{stl}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma^{mnp_2}\theta)(\theta\gamma^{m_2n_2p_2}\theta) > \\
& \qquad - < (\lambda\gamma^{stl}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma^{n}\theta)(\theta\gamma^{m_2n_2m}\theta) > \\
& \qquad + < (\lambda\gamma^{stl}\theta)(\lambda\gamma^{m_1}\theta)(\lambda\gamma^{m}\theta)(\theta\gamma^{m_2n_2n}\theta) > \Big) \qquad \text{(C.9)}
\end{aligned}
$$

Computation based on the Mathematica package, Gamma.m, yields for the first term of (C.8)

$$
= -\frac{1}{3780}\zeta_1^\mu\zeta_2^\nu k_2^\rho e_{3\mu\nu\rho}\, k_3^2 \qquad \text{(C.10)}
$$

Similarly, the second term in (C.8) gives

$$
\begin{aligned}
& \zeta_1^{n_1}\zeta_2^{m_2}e_3^{stn}k_1^{m_1}k_3^l k_3^m < (\lambda\gamma^{st}\gamma^l\theta)(\lambda\gamma^{p_1}\theta)(\lambda\gamma^{mn}\gamma^{m_2}\theta)(\theta\gamma^{m_1n_1p_1}\theta) > \\
= \quad & -\frac{11}{1260}\zeta_1^\mu\zeta_2^\nu k_1^\rho e_{3\mu\nu\rho}\, k_3^2 \qquad \text{(C.11)}
\end{aligned}
$$

These results combine implies for the first term of (C.7) that

$$
\frac{1}{8} < (\lambda A^{(1)})(\lambda\gamma_{mn}A^{(2)})(\lambda\gamma_{st}\gamma^l\theta)k_3^l k_3^m e_3^{stn} > \doteq i\,\zeta_1^\mu\zeta_2^\nu k_2^\rho e_{3\mu\nu\rho}\, k_3^2 \qquad \text{(C.12)}
$$

Following the similar steps, one can show that (after some tedious algebra) the second term of (C.7) yields

$$
\begin{aligned}
& \frac{1}{8}\frac{3}{7} < (\lambda A^{(1)})(\lambda\gamma_{mn}A^{(2)})(\lambda\gamma_{qm}\gamma_{st}\gamma^l\theta)k_3^q k_3^l e^{stn} > \\
\doteq \quad & i\,\zeta_1^\mu\zeta_2^\nu k_1^\rho e_{3\mu\nu\rho}\, k_3^2 \qquad \text{(C.13)}
\end{aligned}
$$

where $\doteq$ indicates that the overall numerical coefficient is not recorded precisely.

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